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Chapter 3

Capital Growth Theory

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1. Introduction

Even casual observation strongly suggests that capital growth is not just a catch-phrase but something which many actively strive to achieve. It is therefore rather surprising that capital growth *theory* is a relatively obscure subject. For example, the great bulk of today's MBA's have had little or no exposure to the subject, having had their attention focussed almost exclusively on the single-period mean–variance model of portfolio choice. The purpose of this essay is to review the theory of capital growth, in particular the so-called growth-optimal investment strategy, its properties, its uses, and its links to betting and other investment models. We also discuss several applications that have tended to refine the basic theory.

The central feature of the growth-optimal investment strategy, also known as the geometric mean model and the Kelly criterion, is the logarithmic shape of the objective function. But the power and durability of the model is due to a remarkable set of properties. Some of these are unique to the growth-optimal strategy and the others are shared by all the members of the (remarkable) small family to which the growth optimal strategy belongs.

Investment over time is multiplicative, not additive, due to the compounding nature of the process itself. This makes a number of results in dynamic investment theory appear nonintuitive. For example, in the single-period portfolio problem, the optimal investment policy is very sensitive to the utility function being used; the set of policies that are inadmissible or dominated across all utility functions is relatively small. The same observation holds in the dynamic case when the number of periods is not large. But as the number of periods does become large, the set of investment policies that are optimal for *current* investment tends to shrink drastically, at least in the basic reinvestment case without transaction costs. As we will see, many strikingly different investors will, in essence, invest the same way when the horizon is distant and will only begin to part company as their horizons near.

It is tempting to conjecture that all long-run investment policies to which risk-averse investors with monotone increasing utility functions will flock, under a favorable return structure, insure growth of capital with a very high probability. Such a conjecture is false; many investors will, even in this case, converge on investment policies which almost surely risk ruin in the long run, in effect ignoring feasible policies which almost surely lead to capital growth. Similarly, the relationship between the behavior of capital over time and the behavior of the expected utility of that same capital over time often appears strikingly nonintuitive.

Section 2 reviews the origins of the capital growth model while Section 3 contains a derivation and identifies its key properties. The conditions for capital growth are examined in Section 4. The model's relationship to other long-run investment models is studied in Section 5 and Section 6 contains its role in intertemporal investment/consumption models. Section 7 adds various constraints for accomplishing tradeoffs between growth and security, while Section 8 reviews various applications. A concluding summary is given in Section 9.

2. Origins of the model

The approach to investment commonly known as the growth-optimal investment strategy has a number of apparently independent origins. In particular, Williams [1936], Kelly [1956], Latane [1959], and Breiman [1960, 1961] seem to have been unaware of each other's papers. But one can also argue that Bernoulli (1738) unwittingly stumbled on it in 1738 in his resolution of the St. Petersburg Paradox — see the 1954 translation — and Samuelson's survey [1977].

Samuelson [1971] appears to be the earliest to have related the geometric mean criterion to utility theory — and to find it wanting. The growth optimal strategy's inviolability in the larger consumption–investment context when preferences for consumption are logarithmic was first noted by Hakansson [1970]. Finally, models considering tradeoffs between capital growth and security appear to have been pioneered by MacLean & Ziemba [1986].

3. The model and its basic properties

The following notation and basic assumptions will be employed:

w_t = amount of investment capital at decision point t (the end of the tth period);

M_t = the number of investment opportunities available in period t, where $M_t \leq M$;

S_t = the subset of investment opportunities which it is possible to sell short in period t;

r_{1t} = rate of interest in period t;

r_{it} = return per unit of capital invested in opportunity i, where $i = 2, \ldots, M_t$, in the tth period (random variable). That is, if we invest an amount θ in i at the beginning of the period, we will obtain $(1 + r_{it})\theta$ at the end of that period;

z_{1t} = amount lent in period t (negative z_{1t} indicate borrowing) (decision variable);

z_{it} = amount invested in opportunity $i, i = 2, \ldots, M_t$ at the beginning of the tth period (decision variable);

$F_t(y_2, y_3, \ldots, y_{M_t}) \equiv \Pr\{r_{2t} \le y_2, r_{3t} \le y_3, \ldots, r_{M_t t} \le y_{M_t}\}$;

$z_t \equiv (z_{2t}, \ldots, z_{M_t t})$;

$x_{it} \equiv \dfrac{z_{it}}{w_{t-1}}, \qquad i = 1, \ldots, M_t$;

$x_t \equiv x_{2t}, \ldots, x_{M_t t}$;

$\langle x_t \rangle \equiv x_1, \ldots, x_t$.

The capital market will generally be assumed to be perfect, i.e. that there are no transaction costs or taxes, that the investor has no influence on prices or returns, that the amount invested can be any real number, and that the investor has full use of the proceeds from any short sale.

The following basic properties of returns will be assumed:

$$r_{1t} \ge 0, \qquad t = 1, 2, \ldots \tag{1}$$

$$E[r_{it}] \ge \delta + r_{1t}, \qquad \delta > 0, \qquad \text{some } i, t = 1, 2, \ldots \tag{2}$$

$$E[r_{it}] \le K, \qquad \text{all } i, t. \tag{3}$$

These assumptions imply that the financial market provides a 'favorable game.'

We also assume that the (nonstationary) return distributions F_t are either independent from period to period or obey a Markov process and they also satisfy the 'no-easy-money condition'

$$\mathrm{P}\left\{\sum_{i=2}^{M_t}(r_{it} - r_{1t})\theta_i < \delta_1\right\} > \delta_2 \text{ for all } t \text{ and all } \theta_i \text{ such that } \sum_{i=2}^{M_t}|\theta_i| = 1, \text{ and } \theta_i \ge 0 \text{ for all } i \notin S_t, \tag{4}$$

where $\delta_1 < 0$, $\delta_2 > 0$.

Condition (4) is equivalent to what is often referred to as the no-arbitrage condition. It is generally a necessary condition for the portfolio problem to have a solution.

We also assume that the investor must remain solvent in each period, i.e., that he or she must satisfy the solvency constraints

$$\Pr\{w_t \ge 0\} = 1, \qquad t = 1, 2, \ldots. \tag{5}$$

The amount invested at time $t-1$ is

$$\sum_{i=1}^{M_t} z_{it} = w_{t-1}$$

and the value of the investment at time t, broken down between its risky and riskfree components, is

$$w_t = \sum_{i=2}^{M_t} (1+r_{it}) z_{it} + (1+r_{1t})\Big(1 - \sum_{i=2}^{M_t} z_{it}\Big),$$

which together yield the basic difference equation

$$\begin{aligned} w_t &= \sum_{i=2}^{M_t} (r_{it} - r_{1t}) z_{it} + w_{t-1}(1+r_{1t}), \qquad t = 1, 2, \ldots \\ &= w_{t-1} R_t(x_t) = w_0 R_1(x_1) \ldots R_t(x_t), \qquad t = 1, 2, \ldots \end{aligned} \tag{6}$$

where

$$R_t(x_t) \equiv \sum_{i=2}^{M_t} (r_{it} - r_{1t}) x_{it} + 1 + r_{1t}. \tag{7}$$

Let us now turn to the basic reinvestment problem which (ignores capital infusions and distributions and) simply revises the portfolio at discrete points in time. In view of (5), (6) may be written

$$w_t = w_0 \exp\Big\{\sum_{n=1}^{t} \ln R_n(x_n)\Big\}, \qquad t = 1, 2, \ldots. \tag{8}$$

Defining

$$G_t(\langle x_t \rangle) \equiv \frac{\Big[\sum_{n=1}^{t} \ln R_n(x_n)\Big]}{t}, \tag{9}$$

(8) becomes

$$\begin{aligned} w_t &= w_0[\exp\{G_t(\langle x_t \rangle)\}]^t \\ &= w_0(1+g_t)^t, \end{aligned} \tag{10}$$

where $g_t = \exp G_t(\langle x_t \rangle) - 1$ is the compound growth rate of capital over the first t periods.

By the law of large numbers,

$$G_t(\langle x_t \rangle) \to E[G_t(\langle x_t \rangle)]$$

under mild conditions. Thus, it is evident that for large T,

$$w_t \to 0 \text{ if } E[G_t] \le \delta < 0, \quad t \ge T, \tag{11}$$

$$w_t \to \infty \text{ if } E[G_t] \geq \delta > 0, \quad t \geq T \tag{12}$$

$$g_t \to \exp E[G_t] - 1. \tag{13}$$

Under stationary returns and policies$\langle x_t \rangle$, (11) and (12) simplify to

$$\left.\begin{aligned} w_t &\to 0 \quad \text{if } E[\ln R_n(x_n)] < 0 \\ w_t &\to \infty \text{ if } E[\ln R_n(x_n)] > 0 \end{aligned}\right\} \quad \text{any } n.$$

There is nothing intuitive that would suggest that the sign of $E[\ln R_n(x_n)]$ is the determinant of whether our capital will decline or grow in the (stationary) simple reinvestment problem. What *is* evident is that the *expected* return on capital, $E[R_n] - 1$, is *not* what matters. As (6) reminds us, capital growth (positive or negative) is a multiplicative, not an additive process.

To illustrate the point, consider the case of only two assets, one riskfree yielding 5% per period, and the other returning either −60% or +100% with equal probabilities in each period. Always putting all of our capital in the riskfree asset clearly gives a 5% growth rate of capital. The *expected* return on the risky asset is 20% per period. Yet placing all of our funds in the risky asset at the beginning of each period results in a capital growth rate that converges to −10.55%! It is easy to see this. We will double our money to 200% roughly half of the time. But we will also lose 60% (bringing the 200% to 80%) of our beginning-of-period capital about half the time, for a 'two-period return' of −20% on average, or −10.55% per period. *Expected* capital $E[w_t]$, on the other hand, has a growth rate of 20% per period.

What this simple example demonstrates is that there are many investment strategies for which, as $t \to \infty$,

$$E[w_t] \to \infty$$

$$\text{Median}\,[w_t] \to 0$$

$$\text{Mode}\,[w_t] \to 0$$

$$\Pr\{w_t < \$1\} \to 1.$$

The coexistence of the above four measures results when $E[G_t] \leq \delta < 0$ for $t \geq T$ and a long (but thin) upper tail is generated as w_t moves forward in time.

In view of (7), (9) and (10), we observe that to 'maximize' the long-run growth rate g_t, it is necessary and sufficient to maximize $E[G_t(\langle x_t \rangle)]$, or

$$\text{Max}\,\{E[\ln R_1(x_1)] + E[\ln R_2(x_2)] + \ldots\} \tag{14}$$

Whenever returns are independent from period to period or the economy obeys a Markov process [1], it is necessary and sufficient to accomplish (14) to

$$\underset{x_t}{\text{Max}}\; E[\ln R_t(x_t)] \text{ sequentially at each } t-1. \tag{15}$$

[1] Algoet & Cover [1988] show formally that the growth-optimal strategy maintains its basic properties under arbitrary returns processes.

Since the geometric mean of $R_t(x_t) = \exp\{E[\ln R_t(x_t)]\}$, we observe that (15) is also equivalent to maximizing the geometric mean of principal plus return at each point in time.

3.1. Properties of the growth-optimal investment strategy

Since the solution $\langle x_t^* \rangle$ to (15), in view of (10) and (13), *almost surely* leads to more capital in the long run than any other investment policy which does not converge to it, $\langle x_t^* \rangle$ is referred to as the growth-optimal investment strategy. Existence is assured by the no-easy-money condition (4), the bounds on expected returns (1)–(3), and the solvency constraint (5). The strict concavity of the objective function in (15) implies that the optimal payoff distribution $R_t(x_t^*)$ is unique; the optimal policy x_t^* itself will be unique only if, for any security i, there is no portfolio of the other assets which can replicate the return pattern r_{it}.

It is probably not surprising that the growth-optimal strategy never risks ruin, i.e.

$$\Pr\{R_t(x_t^*) = 0\} = 0$$

— because to grow you have to survive. But this need not mean that the solvency constraint is not binding: $E[\ln R_t(x_t)]$ may exist even when R_t touches 0 as long as the lower tail is very thin. The conditions (1)–(3) imply that positive growth is feasible. Another dimension of the consistency between short-term and long-term performance was observed by Bell & Cover [1988].

As shown by Breiman [1961], the growth-optimal strategy also has the property that it asymptotically minimizes the expected time to reach a given level of capital. This is not surprising in view of the characteristics noted in the previous two paragraphs.

It is also evident from (15) that the growth-optimal strategy is *myopic* even when returns obey a Markov process (Hakansson 1971c). This property is clearly of great practical significance since it means that the investor only needs to estimate the coming period's (joint) return structure in order to behave optimally in a long-run sense; future periods' return structures have no influence on the current period's optimal decision. *No* other dynamic investment model has this property in a Markov economy; only a small set of other families have it when returns are independent from period to period (see Section 5).

The growth-optimal strategy implies, and is implied by, logarithmic utility of wealth at the end of each period. This is because at each $t-1$

$$\begin{aligned}
&\operatorname*{Max}_{x_t}\ E[\ln R_t(x_t)] \\
&\sim \operatorname*{Max}_{x_t}\{E[\ln R_t(x_t) + \ln w_{t-1}]\} \\
&= \operatorname*{Max}_{x_t}\ E[\ln(w_{t-1}R_t(x_t))] = \operatorname*{Max}_{z_t}\ E[\ln w_t(z_t)].
\end{aligned}$$

Since every utility function is unique (up to a positive linear transformation), it also follows that the growth-optimal strategy is *not* consistent with any other end-of-period utility function (more on this in the next subsection).

The relative risk aversion function

$$q(w) \equiv -\frac{wu''(w)}{u'(w)}$$

equals 1 when $u(w) = \ln(w)$ (it is 0 for a risk-neutral investor). Thus, we observe that to do 'the best' in the long run in terms of capital growth, it is not only necessary to be risk averse in each period. We must also display the 'right' amount of risk aversion. The long-run growth rate of capital will be lower either if one invests in a way which is *more* risk averse than the logarithmic function or relies on an objective function which is *less* risk averse.

The growth-optimal investment strategy is not only linear in beginning-of-period wealth but proportional as well since definitionally

$$z_t^* = w_{t-1}x_t^*.$$

Both of these properties are shared by only a small family of investment models.

Since the growth-optimal strategy is consistent with a logarithmic end-of-period utility function only, it is clearly *not* consistent with the mean–variance approach to portfolio choice — which in turn is consistent with quadratic utility for arbitrary security return structures, and, for normally distributed returns, with those utility functions whose expected utilities exist when integrated with the normal distribution, plus a few other cases, as shown by Ziemba & Vickson [1975] and Chamberlain [1983]. This incompatibility is easy to understand; in solving for the growth-optimal strategy, *all* of the moments of the return distributions matter, with positive skewness being particularly favored. When the returns on the risky assets are normally distributed, no matter how favorable the means and variances are, the growth-optimal strategy cooly places 100% of the investable funds in the riskfree asset.

The preceding does not imply that the growth-optimal portfolio necessarily is far from the mean–variance efficient frontier (although this *may* be the case [see e.g. Hakansson, 1971a]). It will generally be close to the MV-efficient frontier, especially when returns are fairly symmetric. And as shown in Section 8, the mean–variance model can in some cases be used to (sequentially) generate a close approximation to the growth-optimal portfolios.

Other properties of the Kelly criterion can be found in MacLean, Ziemba & Blazenko [1992, table 1].

3.2. Capital growth vs. expected utility

Based on (10), the uniqueness properties implied by (15), and the law of large numbers, it is undisputable, as noted in the previous subsection, that the growth-optimal strategy almost surely generates more capital (under basic reinvestment) in the long run than any other strategy which does not converge to it. At the same time, however, we observed that the growth-optimal strategy is consistent with logarithmic end-of-period utility of wealth *only*. This clearly implies that there must be 'reasonable' utility functions which value almost surely less capital in the long run more than they value the distribution generated by the Kelly criterion.

Consider the family

$$u(w) = \frac{1}{\gamma} w^{\gamma}, \qquad \gamma < 1, \tag{16}$$

to which $u(w) = \ln(w)$ belongs via $\gamma = 0$, and let $\langle x_t(\gamma) \rangle$ be the optimal portfolio sequence generated by solving

$$\operatorname*{Max}_{x_t} E\left[\frac{1}{\gamma} w_t^{\gamma}\right] \text{ at each } t-1.$$

For simplicity, consider the case of stationary returns. Since $x_t(\gamma) \neq x_t(0) = x_t^*$, it is evident that

$$\text{Max } E\left[\frac{1}{\gamma} w_t(\langle x_t(\gamma)\rangle)^{\gamma}\right] > E\left[\frac{1}{\gamma} w_t(\langle x_t^* \rangle)^{\gamma}\right], \quad \gamma \neq 0 \tag{17}$$

even though there exist numbers $a > 1$ and $T(\epsilon)$ such that

$$\Pr\{w_t(\langle x_t(\gamma)\rangle) < w_0 a^t < w_t(\langle x_t^* \rangle)\} \geq 1 - \epsilon, \quad t > T(\epsilon) \tag{18}$$

for every $(1 >)\ \epsilon > 0$.

Many a student of investment has stubbed his toe by interpreting (18) to mean that $\langle x_t^* \rangle$ generates higher expected utility than, say, $\langle x_t(\gamma) \rangle$. (17) and (18) may seem like a paradox but clearly implies that the geometric mean criterion *does not* give rise to a 'universally best' investment strategy.

The intuition behind this truth is as follows. For $\gamma < 0$ in (16), (17) and (18) occur because, despite the fact that the wealth distribution for $\langle x_t(\gamma) \rangle$ lies almost entirely to the *left* of the wealth distribution for $\langle x_t^* \rangle$, the lower tail of the distribution for $\langle x_t(\gamma) \rangle$ is shorter and (imperceptibly) thinner than the (bounded) left tail of the growth-optimal distribution. Thus, for negative powers, very small adverse changes in the lower tail overpower the value of almost surely ending up with a higher compound return. Conversely, for $\gamma > 0$, it is the *longer* (though admittedly very thin) upper tail that gives rise to (17) in the presence of (18) even though, again, the wealth distribution for $\langle x_t(\gamma) \rangle$ lies almost entirely to the *left* of the wealth distribution for $\langle x_t^* \rangle$.

4. Conditions for capital growth

As already noted, the determinants of whether capital will grow or decline (almost surely) in the long run are given by (12) and (11). Conditions (1)–(2) insure that (12) is feasible; in the absence of (1)–(2), positive growth may be infeasible. If a positive long-run growth rate (bounded away from zero) is achievable, then the growth-optimal strategy will find it. Thus we can state:

Theorem. *In the absence of (1) and (2), a necessary and sufficient condition for long-run capital growth to be feasible is that the growth-optimal strategy achieves a positive growth rate, i.e. that for some* $\epsilon > 0$ *and large* T

$$E[\ln R_t(x_t^*)] \geq \epsilon, \quad t \geq T \tag{19}$$

For $\gamma < 0$, the objective functions in (16) attain long-run growth rates of capital between those of the risk-free asset and of the growth-optimal strategy. But for $\gamma > 0$, the long run growth-rate may be negative. Consider for a moment the utility function $u(w) = w^{1/2}$, one of the most frequently cited examples of 'substantial' risk aversion since Bernoulli's time. Even this venerable function may, however, lead to (almost sure) ruin in the long-run: suppose, for example, that the riskfree asset yields 2% per period and that there is only one risky asset, which gives either a loss of 8.2% with probability 0.9, or a gain of 206% with probability 0.1. The optimal policy then calls for investing the fraction 1.5792 in the risky asset (by borrowing the fraction 0.5792 of current wealth to complete the financing) in each period. But the average compound growth rate g_t in (10) will now tend to -0.00756, or $-3/4\%$. Thus, expected utility 'grows' as capital itself almost surely vanishes.

What this example illustrates is that risk aversion plus a favorable return structure [see (1)–(3)] are *not* sufficient to insure capital growth in the basic reinvestment case.

5. Relationship to other long-run investment models

As shown in Section 3, the growth-optimal investment strategy has its traditional origin in arguments concerning capital growth and the law of large numbers. But it can also be derived strictly from an expected utility perspective — but only as a member of a small family.

Let n be the number of periods left to a terminal horizon point at time 0. Assume that wealth at that point, w_0, has utility $U_0(w_0)$, where $U_0' > 0$ everywhere and $U_0'' < 0$ for large w_0. Then, with one period to go, we have the single-period portfolio problem

$$U_1(w_1) \equiv \underset{z_1 | w_1}{\text{Max}} \; E[U_0(w_0(z_1))]$$

where $U_1(w_1)$ is the induced, or derived, utility of wealth w_1 at time 1 and the difference equation (6) has been trivially modified to

$$w_{n-1} = \sum_{i=2}^{M_n} (1 + r_{in}) z_{in} + w_n(1 + r_{1n}), \quad n = 1, 2, \ldots . \tag{20}$$

Thus, with n periods to go, we obtain

$$U_n(w_n) \equiv \underset{z_n | w_n}{\text{Max}} \; E[U_{n-1}(w_{n-1}(z_n))], \quad n = 1, 2, \ldots \tag{21}$$

where (21) is a standard recursive equation.

The induced utility of current wealth, $U_n(w_n)$, of course, generally depends on all the inputs to the problem, that is the utility of terminal wealth U_0, the joint distribution functions of future returns $F_n, \dots, F_1$, and the future interest rates $r_{1n}, \dots, r_{11}$. But there are two rather interesting special cases. The first is the case in which the induced utility functions $U_n(w_n)$ depend *only* on the terminal utility function U_0. This occurs when the returns are independent from period to period and $U_0(w_0)$ is isoelastic, i.e.

$$U_0(w_0) = \frac{1}{\gamma} w_0^{\gamma}, \text{ some } \gamma < 1.$$

As first shown by Mossin [1968], (21) now gives

$$\begin{aligned} U_n(w_n) &= a_n U_0(w_n) + b_n \\ &\sim U_0(w_n) \end{aligned}$$

(where $\sim$ means equivalent to) since a_n and b_n are constraints with a_n positive. The optimal investment policy is both myopic and proportional, i.e.

$$z_{in}^*(w_n) = x_{in}(\gamma) w_n, \text{ all } i$$

where the $x_{in}(\gamma)$ are constants.

The second special case obtains when returns are independent from period to period, interest rates are deterministic, and the terminal utility function reflects hyperbolic absolute risk aversion, that is (Hakansson 1971c)

$$U_0(w_0) = \begin{cases} \dfrac{1}{\gamma}(w_0 + \phi)^{\gamma}, & \gamma < 1, \\ \text{or} & \\ (\phi - w_0)^{\gamma}, & \gamma > 1, \ \phi \text{ large}; \\ \text{or} & \\ -\exp\{-\phi w_0\} & \phi > 0. \end{cases} \tag{22}$$

In the first subcase

$$U_n(w_n) = \frac{1}{\gamma}\left(w_n + \frac{\phi}{(1 + r_1) \dots (1 + r_{1n})}\right)^{\gamma} \tag{23}$$

where (23) holds globally for $\phi \le 0$ and locally for $\phi > 0$, i.e. for $w_n \ge L_n > 0$. The optimal investment policy is

$$z_{in}^*(w_n) = x_{in}(\gamma)\left(w_n + \frac{\phi}{(1 + r_{11}) \dots (1 + r_{1n})}\right), \quad i \ge 2.$$

In the other two subcases, a closed form solution holds only locally.

But the most interesting result associated with (21) is surprisingly general. Under mild conditions on $U_0(w_0)$, and independent (but nonstationary) returns from period to period, we obtain [Hakansson, 1974; see also Leland, 1972; Ross, 1974; Huberman and Ross 1983]:

$$u_n(w_n) \to \frac{1}{\gamma} w_n^{\gamma} \tag{24}$$

and, if returns are stationary,

$$z_{in}^*(w_n) \to x_{in}^*(\gamma) w_n. \tag{25}$$

Thus, the class of utility functions

$$u(w) = \frac{1}{\gamma} w^{\gamma}, \quad \gamma < 1, \tag{16}$$

the only family with constant relative risk aversion (ranging from 0 to infinity) and exhibiting myopic and proportional investment policies, is evidently applicable to a *large* class of long-run investors. The optimal policies above are not mean–variance efficient, but for reasonably symmetric return distributions, they come close to MV efficiency.

Since $\gamma = 0$ in (24) corresponds to logarithmic utility of wealth, the growth-optimal strategy is clearly a member of this elite family of long-run oriented investors. In other words, the geometric mean investment strategy has a solid foundation in utility theory as well.

6. Relationship to intertemporal consumption–investment models

Up to this point, we have examined the basic dynamic investment problem, i.e. without reference to cash inflows or outflows. Under some conditions, the inclusion of these factors is straightforward and does not materially affect the optimal investment policy. But a realistic model incorporating noncapital in- and outflows typically complicates the model substantially.

The basic dynamic consumption–investment model incorporates consumption and a labor income into the dynamic reinvestment model. Following Fisher [1936], wealth is viewed as a means to an end, namely consumption.

The basic difference equation (6) now becomes

$$w_t = \sum_{i=2}^{M_t} (r_{it} - r_{1t}) z_{it} + (1 + r_{1t})(w_{t-1} - c_t) + y_t, \quad t = 1, \ldots, T, \tag{26}$$

where c_t is the amount consumed in period t (set aside at the beginning of the period) and y_t is the labor income received at the end of period t.

Consistent with the foregoing, the individual's objective becomes

$$\text{Max } E[U(c_1, \ldots, c_T)]$$

subject to

$$c_t \geq 0, \text{ all } t$$

where U is assumed to be monotone, strictly concave, and to reflect impatience, i.e. considering the two consumption streams

$$(a,\ b,\ c_3, \ldots, c_T)$$
$$(b,\ a,\ c_3, \ldots, c_T), \quad a > b$$

the first is preferred to the second.

In order to attain tractability, several strong assumptions are usually imposed:

1) the individual's lifetime (horizon) is known,
2) interest rates are viewed as deterministic,
3) the labor income y_t is deterministic; its present value is thus

$$Y_{t-1} \equiv \frac{y_t}{r_{1t}} + \ldots + \frac{y_T}{(1+r_{1t})\ldots(1+r_{1T})},$$

4) the utility function is assumed to be additive, i.e.

$$U(c_1, \ldots, c_T) = u_1(c_1) + \alpha_1 u_2(c_2) + \ldots + \alpha_1 \ldots \alpha_{T-1} u_T(c_T), \tag{27}$$

where $u_t' > 0$, $u_t'' < 0$, and typically $\alpha_t < 1$, for all t, which implies that preferences are independent of past consumption.

Let

$$f_{t-1}(w_{t-1}) \ = \ \text{maximum expected utility at } t-1 \text{ given } w_{t-1}.$$

This gives

$$f_{t-1}(w_{t-1}) = \operatorname*{Max}_{c_t, z_t} \{u_t(c_t) + \alpha_t E[f_t(w_t)]\}, \quad t = 1, \ldots, T, \tag{28}$$

where $f_T(w_T) \equiv 0$ or $b_T(w_T)$

$$\text{subject to} \qquad c_t \geq 0 \tag{29}$$
$$\Pr\{w_t \geq -Y_t\} = 1 \tag{30}$$
$$z_{it} \geq 0, \quad i \notin S_t \tag{31}$$

for each t, where $b_T(w_T)$ represents a possible bequest motive. It is apparent that $f_{t-1}(w_{t-1})$ represents the utility of wealth and that it is *induced* or *derived*; it clearly depends on everything in the model. Solving (28) recursively, it is evident that, under our assumptions concerning labor income and interest rates, Y_t can be exchanged for cash in the solution.

Suppose that in (27)

$$u_t(c_t) = \frac{1}{\gamma} c_t^{\gamma}, \quad \gamma < 1,\ t = 1, \ldots, T. \tag{32}$$

Then [Hakansson, 1970]

$$f_{t-1}(w_{t-1}) = A_{t-1}(w_{t-1} + Y_{t-1})^{\gamma} + B_{t-1},$$
$$c_t^*(w_{t-1}) = C_t(w_{t-1} + Y_{t-1}),$$

$$z_{it}^*(w_{t-1}) = (1 - C_t)x_{it\gamma}^*(w_{t-1} + Y_{t-1}), \quad i \geq 2,$$

and

$$z_{1t}^*(w_{t-1}) = w_{t-1} - c_t^* - \sum_{i=2}^{M_t} z_{it}^*(w_{t-1}),$$

where the A_t, B_t, and C_t are constants. Thus, the optimal consumption and investment policies are again proportional, not to w_{t-1} but to $w_{t-1} + Y_{t-1}$. The latter quantity is sometimes referred to as permanent income.

Note that when $\gamma = 0$ in (32), the consumer-investor does indeed employ the growth-optimal strategy to invested funds.

Finally, the model (28)–(31) has been extended in a number of directions, to incorporate a random lifetime, life insurance, a subsistence level constraint on consumption, a Markov process for the economy, and an uncertain income stream from labor — with limited success [see Hakansson 1969, 1971b, 1972; Miller, 1974]. In general, closed-form solutions do not exist when income streams, payment obligations, and interest rates are stochastic. In such cases, multi-stage stochastic programming models are helpful [see e.g. Mulvey & Ziemba, 1995].

7. Growth vs. security

Empirical evidence suggests that the average investor is more risk averse than the growth-optimal investor, with a risk-tolerance corresponding to $\gamma \approx -3$ in (16) [see e.g. Blume & Friend, 1975]. While real-world investors exhibit a wide range of attitudes towards risk, this means that the majority of investors are in effect willing to sacrifice a certain amount of growth in favor of less variability, or greater 'security'.

7.1 The discrete-time case

In view of the convergence results (24) and (25), it is evident that repeated employment of (16) for any $\gamma < 0$ attains an efficient tradeoff between growth and security, as defined above, for the long-run investor. The concept of 'efficiency' is thus employed in a sense analogous to that used in mean–variance analysis.

A number of more direct measures of the sacrifice of growth for security have also been examined. In particular, MacLean, Ziemba & Blazenko [1992] analyzed the tradeoffs based on three growth and three security measures. The three growth measures are:

1. $E(w_t(\langle x_t \rangle))$], the expected wealth level after t periods;
2. $E[g_t]$, the mean compound growth rate over the first t periods;
3. $E[t : w_t(\langle x_t \rangle) \geq y]$, the mean first passage time to reach wealth level y;

while the three security measures are:

4. $\Pr\{w_t(\langle x_t \rangle) \geq y\}$, the probability that wealth level y will be reached in t periods;

5. $\Pr\{w_t(\langle x_t\rangle) \geq b_t,\ t = 1, 2, \ldots\}$, the probability that the investor's wealth is on or above a specified path;

6. $\Pr\{w_t(\langle x_t\rangle) \geq y$ before $w_t(\langle x_t\rangle) \leq b$, where $b < w_0 < y\}$, which includes the probability of doubling before halving.

Tradeoffs were generated via fractional Kelly strategies, i.e. strategies involving (stationary) mixtures of cash and the growth-optimal investment portfolio. Applied to a stationary environment, these strategies were shown to produce effective tradeoffs in that as growth declines, security increases. However, these tradeoffs, while easily computable, are generally not efficient, i.e. do not maximize security for a given (minimum) level of growth. Other comparisons involving the growth-optimal strategy and half Kelly or other strategies may be found in Ziemba & Hausch [1986], Rubinstein [1991], and Aucamp [1993].

7.2. The continuous-time case

Since transaction costs are zero under the perfect market assumption, it is natural to consider shorter and shorter periods between reinvestment decisions. In the limit, reinvestment takes place continuously. Assuming that the returns on risky assets can be described by diffusion processes, we obtain that optimal portfolios are mean–variance efficient in that the instantaneous variance is minimized for a given instantaneous expected return. The intuitive reason for this is that as the trading interval is shortened, the first two moments of the security's return become more and more dominant [see Samuelson, 1970]. The optimal portfolios also exhibit the separation property — as if returns over very short periods were normally distributed. Over any fixed interval, however, payoff distributions are, due to the compounding effect, usually lognormal. In other words, all investors with the same probability assessments, but regardless of risk attitude, invest in only two mutual funds, one of which is riskfree [Merton, 1971]. See also Karatzas, Lehoczky, Sethi and Shreve [1986] and Sethi and Taksar [1988].

In view of the above, it is evident that the tradeoff between growth and security generated by the fractional Kelly strategies in the continuous-time model when the wealth process is lognormal is efficient in a mean–variance sense. Li [1993] has addressed the growth vs. security question for the two asset case while Li & Ziemba [1992] and Dohi, Tanaka, Kaio & Osaki [1994] have done so when there are n risky assets that are jointly lognormally distributed.

8. Applications

8.1. Asset allocation

In view of the myopic property of the optimal investment policy in the dynamic reinvestment problem [see (24) and (25)], it is natural to apply (15) for different values of γ to the problem of choosing investment portfolios over time. In particular, the choice of broad asset categories, also known as the asset allocation problem, lends itself especially well to such treatment. Thus, to implement the growth-optimal strategy, for example, we merely solve (15) subject to relevant constraints (on borrowing when available and on short positions) at the beginning of each period.

To implement the model, it is necessary to estimate the joint distribution function for next period's returns. Since all moments and comoments matter, one way to do this is to employ the joint empirical distribution for the previous n periods. This approach provides a simple and realistic means of generating nonstationary scenarios of the possible outcomes over time. The raw distribution may of course may be modified in any number of ways, for example via Stein estimators [Jorion, 1985, 1986, 1991; Grauer & Hakansson, 1995], an inflation adapter [Hakansson, 1989], or some other method.

Grauer and Hakansson applied the dynamic reinvestment model in a number of settings with up to 16 different risk attitudes γ under both quarterly and annual portfolio revision. In the domestic setting [Grauer & Hakansson, 1982, 1985, 1986], the model was employed to construct and rebalance portfolios composed of U.S. stocks, corporate bonds, government bonds, and a riskfree asset. Borrowing was ruled out in the first article while margin purchases were permitted in the other two. The third article also included small stocks as a separate investment vehicle. On the whole, the growth-optimal strategy lived up to its reputation. On the basis of the empirical probability assessment approach, quarterly rebalancing, and a 32-quarter estimating period applied to 1934–1992, the growth-optimal strategy outperformed all the others — with borrowing permitted, it earned an average annual compound return of nearly 15%.

In Grauer & Hakansson [1987], the model was applied to a global environment by including in the universe the four principal U.S. asset categories and up to fourteen non-U.S. equity and bond categories. The results showed that the gains from including non-U.S. asset classes in the universe were remarkably large (in some cases statistically significant), especially for the highly risk-averse strategies. With leverage permitted and quarterly rebalancing, the geometric mean strategy again came out on top, generating an annual compound return of 27% over the 1970–1986 period. A different study examined the impact from adding three separate real estate investment categories to the universe of available categories [Grauer & Hakansson, 1994b]. Finally, Grauer, Hakansson & Shen [1990] examined the asset allocation problem when the universe of risky assets was composed of twelve equal- and value-weighted industry components of the U.S. stock market.

Mulvey [1993] developed a multi-period model of asset allocation which incorporates transaction costs, including price impact. The objective function is a general concave utility function. A computational version developed by Mulvey & Vladimirou [1992] focused on the isoelastic class of functions in which the objective was to maximize the expected utility of wealth at the end of the planning horizon. This model, like those based on the empirical distribution approach, can handle assets possessing skewed returns, such as options and other derivatives, and can be extended to include liabilities [see Mulvey & Ziemba, 1995]. Based on historical data over the period 1979 to 1988, this research, based on multi-stage stochastic programming, showed that efficiencies could be gained vis-à-vis myopic models in the presence of transaction costs by taking advantage of the network or linear structure of the problem.

Mean–variance approximations. A number of authors have argued that, in the single period case, power function policies can be well approximated by MV policies, e.g. Levy & Markowitz [1979], Pulley [1981, 1983], Kallberg & Ziemba [1979, 1983], and Kroll, Levy & Markowitz [1984]. However, there is an opposing intuition which suggests that the power functions' strong aversion to low returns and bankruptcy will lead them to select portfolios that are not MV-efficient, e.g. Hakansson [1971a] and Grauer [1981, 1986]. It is therefore of interest to know whether the power policies differ from the corresponding MV and quadratic policies when returns are compounded over many periods.

Let μ_{it} be the expected rate of return on security i at time t and σ_{ijt} be the covariance between the returns on securities i and j at time t. Then the MV investment problem is

$$\operatorname*{Max}_{x_t} \{T(1+\mu_t) - \tfrac{1}{2}\sigma_t^2\},$$

subject to the usual constraints. The MV approximation to the power functions in (16) are obtained [Ohlson, 1975; Pulley, 1981] when

$$T = \frac{1}{1-\gamma}.$$

Under certain conditions this result holds exactly in continuous time [see Merton, 1973, 1980].

With quarterly revision, the MV model was found to approximate the exact power function model very well [Grauer & Hakansson, 1993]. But with annual revision, the portfolio compositions and returns earned by the more risk averse power function strategies bore little resemblance to those of the corresponding MV approximations. Quadratic approximations proved even less satisfactory in this case. These results contrast somewhat with those of Kallberg & Ziemba [1983], who in the quadratic case with smaller variances obtained good approximations for horizons up to a whole year [see also MacLean, Ziemba & Blazenko, 1992].

8.2. Growth–security tradeoffs

The growth vs. security model has been applied to four well-known gambling-investment problems: blackjack, horse race wagering, lotto games, and commodity trading with stock index futures. In at least the first three cases, the basic investment situation is unfavorable for the average player. However, systems have been developed that yield a positive expected return. The various applications use a variety of growth and security measures that appear to model each situation well. The size of the optimal investment gamble also varies greatly, from over half to less than one millionth of one's fortune.

Blackjack. By wagering more in favorable situations and less or nothing when the deck is unfavorable, an average weighted edge is about 2%. An approximation to provide insight into the long-run behavior of a player's fortune is to assume that the game is a Bernouili trial with a probability of success equal to 0.51. With a 2% edge, the optimal wager is also 2% of one's fortune. Professional blackjack teams often use a fractional Kelly wagering strategy with the fraction drawn from the interval 0.2 to 0.8. For further discussion, see Gottlieb [1985] and Maclean, Ziemba & Blazenko [1992].

Horseracing. There is considerable evidence supporting the proposition that it is possible to identify races where there is a substantial edge in the bettor's favor (see the survey by Hausch & Ziemba [1995] in this volume). At thoroughbred racetracks, one can find about 2–4 profitable wagers with an edge of 10% or more on an average day. These opportunities arise because (1) the public has a distaste for the high probability–low payoff wagers, and (2) the public is unable to properly evaluate the worth of multiple horse place and show and exotic wagers because of their complexity; for example, in a ten-horse race there are 120 possible show finishes, each with a different payoff and chance of occurrence. In this situation, interesting tradeoffs between growth and security arise as well.

The Kentucky Derby represents an interesting special case because of the long distance (1 1/4 miles), the fact that the horses have not previously run this distance, and the fame of the race. Hausch, Bain & Ziemba [1995] tabulated the results from Kelly and half Kelly wagers using the system in Ziemba & Hausch [1987] over the 61-year period 1934–1994. They also report the results from using a filter rule based on the horse's breeding.

Lotto games. Lotteries tend to have very low expected payoffs, typically on the order of 40 to 50%. One way to 'beat' parimutuel games is to wager on unpopular numbers — see Hausch & Ziemba [1995] for a survey. But even when the odds are 'turned' favorable, the optimal Kelly wagers are extremely small and it may take a very long time to reach substantial profits with high probability. Often an initial wealth level in the seven figures is required to justify the purchase of even a single $1 ticket. Comparisons between fractional and full Kelly strategies can be found in MacLean, Ziemba & Blazenko [1992].

Commodity trading. Repeated investments in commodity trades can be modeled as a capital growth problem via suitable modifications for margin requirements, daily mark-to-the-market procedures, and other practical details. An interesting example is the turn-of-the-year effect exhibited by U.S. small stocks in January.

One way to benefit from this anomaly is to take long positions in a small stock index and short positions in large stock indices, because the transaction costs (commissions plus market impact) are less than a tenth of what they would be by transacting in the corresponding basket of securities. Using data from 1976 through January 1987, Clark & Ziemba [1987] calculated that the growth-optimal strategy would invest 74% of one's capital in this opportunity. Hence fractional Kelly strategies are suggested. See also Ziemba [1994].

9. Summary

Capital growth theory is useful in the analysis of many dynamic investment situations, with many attractive properties. In the basic reinvestment case, the growth-optimal investment strategy, also known as the Kelly criterion, almost surely leads to more capital in the long run than any other investment policy which does not converge to it. It never risks ruin, and also has the appealing property that it asymptotically minimizes the expected time to reach a given level of capital. The Kelly criterion implies, and is implied by, logarithmic utility of wealth (only) at the end of each period; thus, its relative risk aversion equals 1, which makes it more risk-tolerant than the average investor. As a result, tradeoffs between growth and security have found application in a rich set of circumstances.

The fact that the growth-optimal investment strategy is proportional to beginning-of-period wealth is of great practical value. But perhaps the most significant property of the Kelly criterion is that it is myopic not only when returns are nonstationary and independent but also when they obey a Markov process. In the dynamic investment model with a given terminal objective function, the growth-optimal strategy is a member of the set to which the optimal policy converges as the horizon becomes more distant. Finally, the Kelly criterion is optimal in many environments in which consumption, noncapital income, and payment obligations are present.

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