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CONVERGENCE TO ISOELASTIC UTILITY AND POLICY IN MULTIPERIOD PORTFOLIO CHOICE*

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This paper considers the problem of the investor who has numerous opportunities for revising his portfolio and whose choices are governed by a utility function defined on 'terminal' wealth, $U_0(x_0)$. Attention is focussed on the behavior of the induced utility functions of intermediate wealth with n periods to go, $U_n(x_n)$, and the associated investment policies. Conditions under which the functions $U_n(x_n)$ will tend to isoelasticity have previously been given by Mossin and by Leland. In this paper, the conditions for convergence are weakened further, to the point where they appear sufficiently broad to encompass perhaps most utility functions of practical interest.

1. Introduction

This paper considers the 'reinvestment problem', i.e., the problem of the investor who has numerous opportunities for revising his portfolio and whose choices are governed by a utility function defined on 'terminal' wealth, $U_0(x_0)$. Attention is focussed on the properties of the induced utility functions of intermediate wealth with n periods to go, $U_n(x_n)$, and the associated investment policies.

Mossin (1968) also studied this problem and found that the functions $U_n(x)$ tend to isoelasticity [i.e., to a function of form $(1/\gamma)x^\gamma$] if the terminal function has linear risk tolerance. These results were extended by Leland (1972) who examined the functions for which the [Arrow–Pratt (1963), (1964)] relative risk aversion of $U_0(x_0)$ converges as $x_0 \to \infty$. In this paper, the conditions for convergence are weakened further, to the point where they appear sufficiently broad to encompass perhaps most utility functions of practical interest. The main result gives upper and lower bounds on $U_0(x_0)$ that guarantee convergence of $U_n(x)$ to a member of the isoelastic class (where $-\infty < \gamma < \infty$), in utility as well as policy space, for a broad class of return distributions. The bounds are such that only the behavior of $U_0(x_0)$ for very large x_0 is of consequence; the

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shape of $U_0(x_0)$ for small x_0 is of no import. The significance of the preceding is that a large group of long-run investors with a sufficiently distant horizon can in fact behave optimally by behaving myopically; this is almost never true if the horizon is not far away [Mossin (1968), Hakansson (1971)]. In addition, due to the separation property of the isoelastic class, one mutual fund for each γ that can, via the preceding bounds, be 'associated' with the set of terminal utility functions is both necessary and, apart from differing probability beliefs, sufficient to serve all of these investors in the economy [see Cass and Stiglitz (1970), Hakansson (1970a)].

In a contemporary paper, Ross (1974) examines the optimality of constant proportion investment policies (under stationary returns) and obtains a number of turnpikes that are also optimal policies for the isoelastic class of utility functions. Our results are consistent in that convergence of the (first period) optimal policy to an isoelastic policy need not imply that the isoelastic policy will be optimal to the end. In fact, as Theorems 2 and 4 show, convergence to an isoelastic policy occurs even when that policy, used throughout, would be infeasible. In other words, if Interstate 80 is a good choice all the way for travellers from the East to the San Francisco Bay region, it may be an equally good route for Los Angeles and Seattle destinations – as far as Salt Lake City.

The paper proceeds as follows. The underlying model is developed in sect. 2, along with some preliminary results. Sect. 3 contains further background results plus a simple but revealing set of sufficient conditions for convergence; these conditions are further illuminated by a pair of corollaries and some examples. The main result can be found in sect. 4. Section 5 considers the kind of conditions which must hold for convergence in general (they appear rather weak but are somewhat lacking in operational value), and sect. 6 contains further discussion.

2. Preliminaries

Aside from the assumptions concerning preferences, the premises employed in portfolio theory are highly standardized. A positive risk-free interest rate at which funds can be both borrowed and lent and the existence of one or more risky investment opportunities with stochastically constant returns to scale is usually postulated. Furthermore, perfect liquidity and divisibility of the assets at each (fixed) decision point and absence of transaction costs and taxes are also implicitly assumed in most instances, along with the opportunity to make short sales. Reinvestment of all proceeds at each decision point is generally also postulated, i.e., withdrawals and capital additions are ruled out. Finally, stochastic independence of returns over time are almost universally presumed. These assumptions will again be employed in this paper; some relaxations will be considered in sect. 6. The following basic notation will be adopted:

x_j = amount of investment capital at decision point j (the beginning of the jth period);

M_j = the number of investment opportunities available in period j, where $M_j \leqq M$;

S_j = the subset of investment opportunities which it is possible to sell short in period j;

$r_j - 1$ = rate of interest in period j;

β_{ij} = proceeds per unit of capital invested in opportunity i, where $i = 2, \ldots, M_j$, in the jth period (random variable); that is, if we invest an amount θ in i at the beginning of the period, we will obtain $\beta_{ij}\theta$ at the end of that period;

z_{1j} = amount lent in period j (negative z_{1j} indicate borrowing) (decision variable);

z_{ij} = amount invested in opportunity i, $i = 2, \ldots, M_j$ at the beginning of the jth period (decision variable).

$$F_j(y_2, y_3, \ldots, y_{M_j}) \equiv \Pr\{\beta_{2j} \leqq y_2, \beta_{3j} \leqq y_3, \ldots, \beta_{M_jj} \leqq y_{M_j}\},$$

$$z_j \equiv (z_{2j}, \ldots, z_{M_jj}),$$

$$v_{ij} \equiv \frac{z_{ij}}{x_j}, \qquad i = 1, \ldots, M_j,$$

$$v_j \equiv (v_{2j}, \ldots, v_{M_jj}),$$

v_{ij} clearly denotes the proportion of capital x_j invested in opportunity i at the beginning of period j.

We assume that the (nonstationary) return distributions F_j are independent and satisfy the boundedness conditions

$$0 \leqq \beta_{ij} \leqq K, \qquad \text{all } i, j, \tag{1}$$

$$E[\beta_{ij}] \geqq r_j + n_2, \text{ where } r_j \geqq n_1 > 1, n_2 > 0, \text{ some } i, \text{ all } j, \tag{2}$$

and the 'no-easy-money condition'

$$\Pr\left\{\sum_{i=2}^{M_j} (\beta_{ij} - r_j)\theta_i < \delta_1\right\} > \delta_2, \quad \text{for all } j \text{ and all } \theta_i \text{ such that } \sum_{i=2}^{M_j} |\theta_i| = 1, \text{ and } \theta_i \geqq 0 \text{ for all } i \notin S_j, \text{ where } \delta_1 < 0, \delta_2 > 0. \tag{3}$$

We also assume that the investor must remain solvent in each period, i.e., that he must satisfy the solvency constraints

$$\Pr\{x_{j+1} \geqq 0\} = 1, \quad j = 1, 2, \ldots . \tag{4}$$

This constraint is necessary to achieve consistency with the standard assumption of risk-free lending and to cope with the logical requirements of a multiperiod model of the reinvestment type.

Since the end-of-period capital position is given by the proceeds from current savings, or the negative of the repayment of current debt plus interest, plus the proceeds from current risky investments, we have

$$x_{j+1} = r_j z_{1j} + \sum_{i=2}^{M_j} \beta_{ij} z_{ij}, \quad j = 1, 2, \ldots,$$

where

$$\sum_{i=1}^{M_j} z_{ij} = x_j, \qquad j = 1, 2, \ldots .$$

Combining the preceding we obtain

$$\begin{aligned} x_{j+1} &= \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j x_j, \quad j = 1, 2, \ldots, \\ &= x_j R_j(v_j), \qquad j = 1, 2, \ldots, \end{aligned} \tag{5}$$

where

$$R_j(v_j) \equiv \sum_{i=2}^{M_j} (\beta_{ij} - r_j) v_{ij} + r_j. \tag{6}$$

Note that $R_j(v_j)$ is 1 plus the return on the whole portfolio v_j in period j. The portfolio problem at decision point j is now seen to be one of choosing the vector of risky investments $z_j \equiv (z_{2j}, \ldots, z_{M_j j})$ (proportions v_j) so as to produce the most 'favorable' distribution of end-of-period capital x_{j+1} {or $R_j(v_j)$}. Clearly,

$$v_{1j} = 1 - \sum_{i=2}^{M_j} v_{ij}. \tag{7}$$

Note that when $x_j > 0$, the solvency constraint (4) is equivalent to the constraint

$$\Pr\{R_j(v_j) \geqq 0\} = 1, \quad j = 1, 2, \ldots . \tag{8}$$

Solving (5) recursively, we obtain

$$x_{N+1} = x_1 \prod_{j=1}^{N} R_j(v_j), \quad N = 1, 2, \ldots . \tag{9}$$

We now give some preliminary results. The following lemmas are trivial generalizations of Lemma 1 in Hakansson (1970b):

Lemma 1. *Let* (1), (2), *and* (3) *hold. Then the sets* V_j *of portfolios* v_j *which satisfy the short-sale constraints*

$$v_{ij} \geqq 0, \quad i \notin S_j, \tag{10}$$

and the solvency constraints (8) *contain risky portfolios in addition to the riskless ones and are closed, uniformly bounded, and convex.*

Corollary 1. *For every feasible portfolio* v_j *with an expected return greater than the interest rate, the variance of return is positive, i.e.,*

$$\text{var}\,[R_j(v_j)] > 0 \quad \textit{if } E[R_j(v_j)] > r_j, \textit{ all } j. \tag{11}$$

Lemma 2. *Let* (1), (2), (3), *and* (5) *hold and let* d *be a real number. Then the sets* $Z_j(x_j, d)$ *of portfolios* z_j *which satisfy the short-sale constraints*

$$z_{ij} \geqq 0, \quad i \notin S_j, \tag{12}$$

and the constraints

$$\Pr\{x_{j+1} \geqq d\} = 1, \tag{13}$$

are non-empty for $x_j \geqq d/r_j$. *Moreover, they contain risky portfolios in addition to the riskless ones and are closed, bounded, and convex; when* $x_j = d/r_j$, *only the riskless portfolio* $z_j = (0, \ldots, 0)$ *satisfies* (12) *and* (13).

The following corollaries are immediate:

Corollary 2. $Z_j(x, d) \subset Z_j(y, d)$ *for* $x < y$ *whenever* $y \geqq d/r_j$, *all* j.

Corollary 3. $Z_j(x_j, d_1) \subset Z_j(x_j, d_2)$ *for* $d_1 > d_2$ *whenever* $d_2 \leqq x_j r_j$, *all* j.

Lemma 3. *Let* f *be a continuous function on* $[(d, \infty)$ *and let* (5) *hold. Then* $g(x_j)$ *given by*

$$g(x_j) \equiv \max_{z_j \in Z_j(x_j, d)} E[f(x_{j+1})]$$

exists and is continuous for $x_j \geqq d/r_j$. *If* f *is monotone increasing, so is* g. *If* f *is strictly concave, so is* g, *and the solution* $z_j^*(x_j, d)$ *is then unique.*

The existence of $g(x_j)$ is insured by the compactness of $Z_j(x_j, d)$ (see Lemma 2), (1), and the continuity of $E[f(x_{j+1})]$ as a function of z_j. The continuity of $g(x_j)$ follows from the additional fact that the boundary of $Z_j(x_j, d)$ is continuous in x_j. Monotonicity derives from the observation that for $\delta > 0$,

$$\begin{aligned} g(x_j+\delta)-g(x_j) &\geqq E[f(x'_{j+1}+r_j\delta)-f(x'_{j+1})], \\ &> 0, \quad \text{if } f'(x) > 0, \end{aligned}$$

where

$$g(x_j) = E[f(x'_{j+1})].$$

For a proof that $g(x_j)$ is strictly concave and the optimal policy $z^*(x_j, d)$ is unique when f is strictly concave, see Fama (1970).

Lemma 4. *Let f and g be continuous and defined for* $x \geqq d_f$ *and* $x \geqq d_g$, *respectively, where* $d_f \leqq d_g$. *Moreover, let* $f \geqq g$ $(f > g)$ *for* $x \geqq d_g$ *and let* (5) *hold. Then* $f_1(x) \geqq g_1(x)$ $(f_1(x) > g_1(x))$ *for* $x \geqq d_g/r_j$, where

$$f_1(x_j) \equiv \max_{z_j \in Z_j(x_j, d_f)} E[f\{x_{j+1}(x_j, z_j)\}], \tag{14}$$

and

$$g_1(x_j) \equiv \max_{z_j \in Z_j(x_j, d_g)} E[g\{x_{j+1}(x_j, z_j)\}]. \tag{15}$$

Proof. By Lemma 3, $f_1(x)$ and $g_1(x)$ exist and are continuous for $x \geqq d_g/r_j$. Let $\bar{z}_j(x_j)$ be the solution to eq. (15). Since $Z_j(x_j, d_g) \subset Z_j(x_j, d_f)$ by Corollary 3, $\bar{z}_j(x_j) \in Z_j(x_j, d_f)$; $f \geqq g$ now gives

$$\begin{aligned} \max_{z_j \in Z_j(x_j, d_g)} E[g\{x_{j+1}(x_j, z_j)\}] &= E[g\{x_{j+1}(x_j, \bar{z}_j)\}], \\ &\leqq E[f\{x_{j+1}(x_j, \bar{z}_j)\}], \\ &\leqq \max_{z_j \in Z_j(x_j, d_f)} E[f\{x_{j+1}(x_j, z_j)\}], \end{aligned}$$

(when $f > g$ the first inequality is strict), which completes the proof.

3. First results

We consider the situation in which the investor, while having the opportunity to make numerous reinvestment decision at discrete points in time, is concerned only with his terminal wealth position, x_0. We assume that his preferences for terminal wealth are representable by a utility function, $U_0(x_0)$, which is continuous,[1] monotone increasing, and defined for $x_0 \geqq c \geqq 0$. Letting $U_n(x_n)$ be the induced utility function, with n periods to go, of wealth x_n we obtain, using the principle of optimality, the recursive relation

$$U_n(x_n) = \max_{z_n \in Z_n[x_n, c/(r_0 \ldots r_{n-1})]} E[U_{n-1}\{x_{n-1}(x_n, z_n, F_n, r_n)\}], \tag{16}$$

$$n = 1, 2, \ldots,$$

where $r_0 \equiv 1$ and $x_{n-1}(x_n, z_n, F_n, r_n)$ is given by (5). By repeated application of Lemma 3, each function $U_n(x_n)$, $n = 1, 2, \ldots,$ exists for $x_n \geqq c/(r_1 \ldots r_n)$ and

[1]This assumption may be viewed as redundant since every von Neumann–Morgenstern utility function defined on an interval is continuous.

is continuous and monotone. Furthermore, should $U_0(x_0)$ be strictly concave, so is $U_n(x_n)$, $n = 1, 2, \ldots$.

It is clear that the induced utility functions $U_1(x_1), \ldots, U_n(x_n)$ depend on $F_1, \ldots, F_n, r_1, \ldots, r_n$ as well as $U_0(x_0)$, at least under assumptions (1)–(3). But it may happen that the influence of the return structure is merely to make $U_1(x), \ldots, U_n(x)$ (positive) linear transformations of $U_0(x)$.[2] This is the case if and only if $U_0(x)$ is isoelastic, i.e., it may be written [Mossin (1968)]

$$U_0(x) = \frac{1}{\gamma}x^\gamma, \tag{17}$$

where $\gamma = 0$ represents $U_0(x) = \log x$.

In what follows, we shall make use of the following more general result:

Theorem 1. *[Mossin* (1968)] *Let* $U_0(x_0)$ *have the form*

$$U_0(x_0) = \frac{1}{\gamma}(x_0+m)^\gamma. \tag{18}$$

Then the induced functions $\bar{U}_1(x_1)$, $\bar{U}_2(x_2), \ldots$ *given by the recursive relations*

$$\bar{U}_n(x_n) = \max_{z_n \in Z_n[x_n,\, -m/(r_0 \ldots r_{n-1})]} E[\bar{U}_{n-1}(x_{n-1})], \qquad n = 1, 2, \ldots, \tag{19}$$

where $\bar{U}_0(x_0) \equiv U_0(x_0)$, *have the form*

$$\begin{aligned}\bar{U}_n(x_n) &= \frac{k_1(\gamma)\ldots k_n(\gamma)}{\gamma}\left(x_n+\frac{m}{r_1\ldots r_n}\right)^\gamma, \qquad n = 1, 2, \ldots, \\ &\sim \frac{1}{\gamma}\left(x_n+\frac{m}{r_1\ldots r_n}\right)^\gamma,\end{aligned} \tag{20}$$

where the constants $k_1(\gamma), \ldots, k_n(\gamma)$ *are given by*

$$k_j(\gamma) \equiv \gamma \max_{v_j \in V_j} E\left[\frac{1}{\gamma}R_j(v_j)^\gamma\right]. \tag{21}$$

Moreover, the solution $\bar{z}_{n\gamma}(x_n)$ *to* (19) *is given by*

$$\bar{z}_{n\gamma}(x_n) = [x_n+m/(r_1\ldots r_n)]v_{n\gamma}, \tag{22}$$

where the vector $v_{j\gamma}$ *is the solution to* (21). (The equivalence, written $\sim$, follows from the fact that $k_1(\gamma), \ldots, k_n(\gamma)$ are positive constants.)

Theorem 1 confirms that when $m = 0$ in (18),

$$\begin{aligned}U_n(x_n) &= k_1(\gamma)\ldots k_n(\gamma)U_0(x_n), \qquad n = 1, 2, \ldots, \\ &\sim U_0(x_n),\end{aligned}$$

[2]When there is no risk of ambiguity, the subscript on x will be dropped.

so that the optimal portfolio policy in any period is myopic and based on the terminal utility function directly. Note also that (19) is identical to (16) when $-m = c$. Since the solvency constraint only requires $c \geqq 0$, and the functions (18) are defined for $x_0 \geqq -m$, it follows that (20) solves (16) whenever $m \leqq 0$, i.e.,

$$U_n(x_n) = \frac{k_1(\gamma)\ldots k_n(\gamma)}{\gamma}\left(x_n + \frac{m}{r_1\ldots r_n}\right)^\gamma, \quad n = 1, 2, \ldots,$$

$$\sim \frac{1}{\gamma}\left(x_n + \frac{m}{r_1\ldots r_n}\right)^\gamma,$$

whenever

$$U_0(x_0) = \frac{1}{\gamma}(x_0+m)^\gamma, \quad m \leqq 0.$$

But when $m > 0$ in (18), (20) is not a valid solution to (16) [Hakansson (1971)], since (20) does not obtain if the solvency constraint (4) (i.e., $c \geqq 0$) must be observed in (19).

Lemma 5. [*Hakansson and Miller* (1973)]. *Let* (1)–(3) *hold and let* $v_{j\gamma}$ *be the solution to* (21). *Then for each* γ *there exists an* $\varepsilon(\gamma)$ *such that*

$$\sum_{i=2}^{M_j} |v_{ij\gamma}| > \varepsilon(\gamma) > 0, \quad j = 1, 2, \ldots.$$

Thus, the optimal portfolios generated by the class of functions (17) under assumptsions (1)–(3) always contain, for each γ, a minimal fraction of risky assets in each period.

We now give an important intermediate result.

Lemma 6. $k_j(\gamma)$ *given by* (21) *is positive and increasing in* γ, *with* $k_j(0) = 1$.

Proof. It is immediate from (2) and Lemma 1 that $k_j(\gamma) > 1$, $\gamma > 0$, $k_j(0) = 1$, $0 < k_j(\gamma) < 1$ for $\gamma < 0$. Thus it suffices to demonstrate that $k_j(\gamma-\delta) < k_j(\gamma)$ for $\delta > 0$ and, whenever γ is positive, $\delta < \gamma$.

Let

$$f(x) \equiv \frac{1}{\gamma}\frac{r_j^\delta}{(1-\delta/\gamma)^{\gamma-\delta}}\left(x - \frac{\delta r_j}{\gamma}\right)^{\gamma-\delta}, \quad x \geqq \frac{\delta r_j}{\gamma},$$

$$g(x) \equiv \frac{1}{\gamma}x^\gamma, \quad x \geqq 0.$$

We obtain

$$f(r_j) = g(r_j) = r_j^\gamma/\gamma, \quad f'(r_j) = g'(r_j) = r_j^{\gamma-1}. \tag{23}$$

We shall first show that

$$f(x) \lesseqgtr g(x), \quad \text{if } \gamma \gtrless 0,\ x \neq r_j, \tag{24}$$

by examining the first derivative of

$$A(x) \equiv \frac{f(x)}{g(x)} = \frac{r_j^\delta}{[1-(\delta/\gamma)]^{\gamma-\delta}} \frac{[x-(\delta r_j/\gamma)]^{\gamma-\delta}}{x^\gamma}, \qquad x \geqq \max\,[0, \delta r_j/\gamma].$$

Differentiating with respect to x,

$$A'(x) = \frac{r_j^\delta x^{-\gamma-1}[x-(\delta r_j/\gamma)]^{\gamma-\delta-1}\,\delta}{[1-(\delta/\gamma)]^{\gamma-\delta}}\,(r_j - x) \gtreqless 0, \quad \text{if } x \lesseqgtr r_j,$$

which, coupled with (23), implies (24).

By Theorem 1,

$$\max_{z_j \in Z_j[x_j, \delta r_j/\gamma]} E[f(x_{j+1})] = \frac{k_j(\gamma-\delta) r_j^\delta}{\gamma[1-(\delta/\gamma)]^{\gamma-\delta}} \left(x_j - \frac{\delta}{\gamma}\right)^{\gamma-\delta},$$

$$x_j \geqq \frac{\delta}{\gamma}, \quad j = 1, 2, \ldots,$$

$$\max_{z_j \in Z_j(x_j, 0)} E[g(x_{j+1})] = [k_j(\gamma)/\gamma] x_j^\gamma, \qquad x_j \geqq 0, j = 1, 2, \ldots.$$

But by (23), (24), and Lemma 4,

$$\frac{k_j(\gamma-\delta) r_j^\delta}{[1-(\delta/\gamma)]^{\gamma-\delta}} \left(x_j - \frac{\delta}{\gamma}\right)^{\gamma-\delta} \leqq k_j(\gamma) x_j^\gamma, \quad x_j \geqq \max\left\{0, \frac{\delta}{\gamma}\right\}. \tag{25}$$

Since $\delta/\gamma < 1$, (25) must hold for $x_j = 1$; inserting $x_j = 1$, (25) gives

$$\frac{k_j(\gamma-\delta)}{k_j(\gamma)} \leqq r_j^{-\delta}. \tag{26}$$

Thus $k(\gamma-\delta) < k(\gamma)$ for $\delta > 0$ (provided $\delta < \gamma$ should γ be positive), which completes the proof.

Corollary 4. *Let $\gamma_1 < \gamma_2$ and $k_1(\gamma_1), k_2(\gamma_1), \ldots$ and $k_1(\gamma_2), k_2(\gamma_2), \ldots$ be given by* (21). *Then there exists a number $k < 1$ such that*

$$\frac{k_j(\gamma_1)}{k_j(\gamma_2)} \leqq k, \quad j = 1, 2, \ldots.$$

Proof. Apply (2) to (26).

We now define

$$U_0(x_0) \equiv a + b u_0(x_0), \tag{27}$$

$$u_n(x_n) \equiv \frac{U_n(x_n)}{k_1(\gamma)\dots k_n(\gamma)}, \quad n = 1, 2, \dots, \tag{28}$$

where $b > 0$ and a are constants and $k_1(\gamma), \dots, k_n(\gamma)$ are given by (21). Since $u_1(x_1)$, $u_2(x_2), \dots$ are merely positive linear transformations of $U_1(x_1)$, $U_2(x_2), \dots$, they too are valid utility functions; for any finite n, $U_n(x_n)$ and $u_n(x_n)$ give the same solution.

Our first result of interest is:

Theorem 2. Let $u_0(x_0)$ be a continuous and monotone increasing (terminal wealth utility) function defined for $x_0 \geqq c \geqq 0$ satisfying[3]

$$\frac{1}{\gamma}(x_0-d)^\gamma - A(\gamma) \leqq U_0(x_0) \leqq \frac{1}{\gamma}(x_0+d)^\gamma + A(\gamma), \quad x_0 \geqq c, \tag{29}$$

for some numbers $d \geqq c$, γ, and $A(\gamma)$, where $U_0(x_0)$ is given by (27) and $A(\gamma)$ is positive if $\gamma > 0$ and 0 otherwise. Then the induced (utility of wealth) functions $u_n(x_n)$ [given by (28), (16), and (21)] satisfy

$$\frac{1}{\gamma}(x_n - D_n)^\gamma - k^n A(\gamma) \leqq u_n(x_n) \leqq \frac{1}{\gamma}(x_n + D_n)^\gamma + k^n A(\gamma),$$

$$x_n \geqq \frac{c}{r_1 \dots r_n}, \quad n = 1, 2, \dots, \tag{30}$$

where $k < 1$ and

$$D_n \equiv \frac{d}{r_1 \dots r_n} \leqq d_{\eta\iota}^{-n} \to 0, \tag{31}$$

i.e.,

$$\frac{1}{\gamma}x^\gamma \leqq \lim u_n(x) \leqq \frac{1}{\gamma}x^\gamma, \quad x \geqq 0.$$

In addition, whenever $U_0(x_0)$ is strictly concave and returns are stationary, there exists, for every $\varepsilon > 0$ and all x on any finite interval bounded away from zero, a number $N(\varepsilon)$ such that the optimal policy $z_n^*(x)$ satisfies

$$\sum_{i=2}^{M_n} |z_{in}^*(x) - v_{in\gamma}x| < \varepsilon, \quad n \geqq N(\varepsilon), \tag{32}$$

where $v_{j\gamma}$ is the solution to (21), i.e., the optimal policy converges to the optimal policy for the utility function $u(x) = (1/\gamma)x^\gamma$.

Proof. Let $B_0(x_0) \equiv (1/\gamma)(x_0-d)^\gamma - A(\gamma)$ for $x_0 \geqq d$, and $A_0(x_0) \equiv$

[3]If $a(x)$ is defined for $x \geqq x_1$ and $b(x)$ for $x \geqq x_2 > x_1$, we will write $a(x) \geqq b(x)$, $x \geqq x_1$, whenever the inequality holds for $x \geqq x_2$.

$(1/\gamma)(x_0+d)^\gamma+A(\gamma)$ for $x_0 \geqq -d$, and define $B_1(x_1)$, $B_2(x_2), \dots$ and $A_1(x_1)$, $A_2(x_2), \dots$ by the relations

$$B_n(x_n) = \max_{z_n \in Z_n(x_n,\, D_{n-1})} E[B_{n-1}\{x_{n-1}(x_n, z_n)\}], \quad n = 1, 2, \dots, \tag{33}$$

$$A_n(x_n) = \max_{z_n \in Z_n(x_n,\, -D_{n-1})} E[A_{n-1}\{x_{n-1}(x_n, z_n)\}], \quad n = 1, 2, \dots. \tag{34}$$

By Theorem 1,

$$B_n(x_n) = k_1(\gamma)\dots k_n(\gamma)\frac{1}{\gamma}(x_n-D_n)^\gamma - A(\gamma), \quad n = 1, 2, \dots, \tag{35}$$

$$A_n(x_n) = k_1(\gamma)\dots k_n(\gamma)\frac{1}{\gamma}(x_n+D_n)^\gamma + A(\gamma), \quad n = 1, 2, \dots; \tag{36}$$

(35) and (36) exist for $x_n \geqq D_n$ and $x_n \geqq -D_n$, respectively.

Letting

$$C_n \equiv \frac{c}{r_1 \dots r_n},$$

we obtain, since $-d < c \leqq d$,

$$-D_n < C_n \leqq D_n, \quad n = 1, 2, \dots, \tag{37}$$

and the functions $U_n(x_n)$, $n = 1, 2, \dots$, given by (16) exist for $x_n \geqq C_n$. Thus by (29), Lemma 4, and (37) (recall footnote 3)

$$B_n(x_n) \leqq U_n(x_n) \leqq A_n(x_n), \quad x_n \geqq C_n,\ n = 1, 2, \dots. \tag{38}$$

By Lemma 6 and Corollary 4 there exists a k such that

$$k_n(\gamma)^{-1} \leqq k < 1, \quad \gamma > 0,\ \eta = 1, 2, \dots. \tag{39}$$

Dividing (38) by $k_1(\gamma)\dots k_n(\gamma)$ and using (28), (35), (36), and (39) gives (30). (31) follows from the fact that $\eta_1 > 1$ [see (2)] by assumption. Thus letting

$$\underline{u}_n(x_n) \equiv \frac{1}{\gamma}(x_n-D_n)^\gamma - \frac{A(\gamma)}{k_1(\gamma)\dots k_n(\gamma)}, \tag{40}$$

$$\bar{u}_n(x_n) \equiv \frac{1}{\gamma}(x_n+D_n)^\gamma + \frac{A(\gamma)}{k_1(\gamma)\dots k_n(\gamma)}, \tag{41}$$

where, by (39),

$$\frac{A(\gamma)}{k_1(\gamma)\dots k_n(\gamma)} \leqq k^n A(\gamma),$$

we obtain

$$\underline{u}_n(x) > \underline{u}_{n-1}(x), \quad n = 1, 2, \dots, \tag{42}$$

$$\bar{u}_n(x) < \bar{u}_{n-1}(x), n = 1, 2, \dots, \tag{43}$$

and taking limits,

$$\frac{1}{\gamma}x^\gamma = \lim \underline{u}_n(x) \leqq \lim u_n(x) \leqq \lim \bar{u}_n(x) = \frac{1}{\gamma}x^\gamma. \tag{44}$$

To prove (32) under stationary returns, denote the solutions to (33), (16), and (34) by $\underline{z}_n(x)$, $z_n^*(x)$, and $\bar{z}_n(x)$, respectively, and let [see (33) and (34)], for $x \geqq D_n$,

$$\bar{Z}_n(x) \equiv \{z: E[A_{n-1}\{x_{n-1}(x, z)\}] \geqq B_n(x)\}, \qquad n = 1, 2, \dots, \tag{45}$$

and

$$\bar{V}_n(x) \equiv \{v: E[A_{n-1}\{x_{n-1}(x, vx)\}] \geqq B_n(x)\}, \quad n = 1, 2, \dots .$$

But we also have that

$$\left\{z: \frac{1}{k(\gamma)}E[\bar{u}_{n-1}\{x_{n-1}(x, z)\}] \geqq \underline{u}_n(x)\right\} = \bar{Z}_n(x), \quad \text{all finite } n, \tag{46}$$

since $A_{n-1}(x)$ and $B_n(x)$ differ from $\bar{u}_{n-1}(x)/k_n(\gamma)$ and $\underline{u}_n(x)$ only by the multiplicative constant $k_1(\gamma)\dots k_n(\gamma)$, which, in the stationary case equals $k(\gamma)^n$. Clearly,

$$\bar{Z}_n(x) \subseteq Z_n(x, -D_{n-1}), \qquad n = 1, 2, \dots,$$

so that $\bar{Z}_n(x)$ is bounded for each x (Lemma 2), and by (38)

$$\underline{z}_n(x), z_n^*(x), \bar{z}_n(x) \in \bar{Z}_n(x), \qquad n = 1, 2, \dots . \tag{47}$$

Thus, $\bar{V}_n(x)$ is bounded for any $x > D_n$ and by (35) and (36)

$$\bar{V}_n(x) \subset \bar{V}_n(y), x > y > D_n, \quad n = 1, 2, \dots . \tag{48}$$

When $u_0(x)$ is strictly concave, so are $U_1(x)$, $U_2(x), \dots, A_1(x), A_2(x), \dots$ by Lemma 3; the solutions $z_1^*(x), z_2^*(x), \dots \bar{z}_1(x), \bar{z}_2(x), \dots$ are then unique and $\gamma < 1$.[4] Since $E[A_{n-1}\{x_{n-1}(x_n, z)\}]$ is strictly concave in z, there is some $n\,(=N)$ such that, in view of (44),

$$\bar{Z}_N(x) \subset Z_N(x, -D_{N-1}), \quad x > D_N.$$

But under stationary returns (42) and (43) now imply [see (46)]

$$\bar{Z}_n(x) \subseteq \bar{Z}_{n-1}(x), \quad n > N, \tag{49}$$

and by (44)

$$\lim_{n\to\infty} \left\{z: \frac{1}{k(\gamma)}E[\bar{u}_{n-1}\{x_{n-1}(x, z)\}] \geqq \underline{u}_n(x)\right\}$$
$$= \left\{z: \frac{1}{k(\gamma)}E\left[\frac{1}{\gamma}\, y(x, z)^\gamma\right] \geqq \frac{1}{\gamma}\, x^\gamma\right\} = v_\gamma x, \tag{50}$$

[4]In this case $B_1(x)$, $B_2(x), \dots$ are also strictly concave but this property is not utilized.

since $z(x) = v_\gamma x$, by Theorem 1 and the fact that $\gamma < 1$, is the unique solution to

$$\max_z E\left[\frac{1}{\gamma} y(x, z)^\gamma\right] = k(\gamma) \frac{1}{\gamma} x^\gamma.$$

Since $M_n \leqq M$ and $z_n^*(x)$ is defined for $x \geqq C_n \to 0$, (45)–(50) now imply (32), which completes the proof.

The following corollaries are immediate:

Corollary 5. *Assume that* $u_0(x)$ *is bounded and satisfies* (29). *Then* $u_n(x) \to (1/\gamma)x^\gamma$, $\gamma < 0$, *i.e., the induced utility functions will tend to lose their lower* (*but not their upper*) *bound.*

Corollary 6. *Assume that* $u_0(x_0)$ *is unbounded and satisfies* (29). *Then* $u_n(x) \to (1/\gamma)x^\gamma$, $\gamma \geqq 0$, *i.e., if* $\gamma > 0$, *the induced utility functions will tend to become bounded below.*

In words, Theorem 2 says that if for a given terminal utility function $U_0(x)$ there *exists* a power function which, shifted any finite distance d to the right (and, if the power is positive also any finite distance downward) becomes a lower bound (on $[d, \infty]$) and, shifted in the opposite direction(s) becomes an upper bound to $U_0(x)$, then the induced utility functions converge to that power function.

Remark 1. $u_0(x_0) = (1/\gamma)x_0^\gamma - A\sum_{i=1}^N x_0^{\gamma_i}$, where γ, $A > 0$, $\gamma_i < 0$, $i = 1, \ldots, N$, satisfies (29); hence $u_n(x_n)$, $n = 1, 2, \ldots$, satisfies (30) and if $\gamma < 1$ and returns are stationary the optimal policy $z_n^*(x)$ satisfies (32).

Remark 2. If $u_0(x_0)$ satisfies (29) for some γ, then it satisfies (29) for that γ alone.

In interpreting Theorem 2, it is clear that the shape of the utility function for low levels of wealth is unimportant; the bounds (29) only restrict $u_0(x_0)$ for very large wealth levels. Thus, for example, any Friedman–Savage (1948) terminal utility function which satisfies (29) for large x_0 always does so for smaller x_0 and would be covered by either Corollary 5 or Corollary 6.

Let $q_0^*(x) \equiv -xu_0''(x)/u_0'(x)$ be the relative risk aversion function for $u_0(x)$. It is readily verified that $q_0^*(x)$ need not converge as x becomes large for terminal utility functions satisfying (29) since $q_0^*(x)$ may oscillate arbitrarily around $1-\gamma$ indefinitely and still satisfy (29). Thus, Theorem 2 goes farther than the results of Leland (1972), for which $q_0^*(x)$ is required to converge. In addition, Theorem 2 was obtained without use of Leland's assumption (3) (1972, pp. 27, 32–33).

How broad is the class of terminal utility functions encompassed by Theorem 2? One way to throw light on this question is to identify the functions which do not satisfy (29). One class of such functions is clearly that for which we can find a γ and a sequence $\{x_i\}$, $x_i \to \infty$, such that for every $d > 0$ [and whenever $\gamma > 0$ every $A(\gamma) > 0$]

$$U_0(x_i') \geqq \frac{1}{\gamma}(x_i'+d)^\gamma + A(\gamma), \qquad U_0(x_i'') \leqq \frac{1}{\gamma}(x_i''-d)^\gamma - A(\gamma), \tag{51}$$

where neither of the (non-overlapping) subsequences $\{x_i'\}$ and $\{x_i''\}$ is finite. (It may be noted that if (51) is true only for $x_i' \leqq x$ or $x'' \leqq x$, larger d [and, if applicable, larger $A(\gamma)$] can be found such that (29) holds.) Another possibility is that for every $\gamma > 0$ there is a number $L(\gamma)$ such that

$$U_0(x) > \frac{1}{\gamma}x^\gamma, \quad x \geqq L(\gamma), \tag{52}$$

or that for every $\gamma < 0$ there is a number $L(\gamma)$ such that

$$0 > U_0(x) > \frac{1}{\gamma}x^\gamma, \quad x \geqq L(\gamma). \tag{53}$$

(52) would be satisfied by utility functions whose relative risk aversion $q_0^*(x) \to -\infty$ and (53) by utility functions whose $q_0^*(x) \to +\infty$ as x increases.

Consider (53). Setting $d = L(\gamma)$ we obtain

$$\frac{1}{\gamma}\{x_0 - d(\gamma)\}^\gamma < U_0(x_0) < 0, \quad \text{any } \gamma < 0,\ x_0 \geqq c.$$

By Lemma 4 and Theorem 1, we obtain [after division by $k_1(\gamma) \ldots k_n(\gamma)$]

$$\frac{1}{\gamma}\{x_n - D_n(\gamma)\}^\gamma < u_n(x_n) < 0. \tag{54}$$

Since $D_n(\gamma)$ satisfies (31) for all γ, no matter how negative, this gives, using analogous reasoning for (52):

Theorem 3. *For every* $\varepsilon > 0$ *and every negative (positive)* γ *there is a number* $N(\varepsilon, \gamma)$ *such that*

(i) *if* $U_0(x)$ *satisfies* (53)

$$\frac{1}{\gamma}(x-\varepsilon)^\gamma < u_n(x) < 0, \quad n \geqq N(\varepsilon, \gamma), \tag{55}$$

(ii) *if* $U_0(x)$ *satisfies* (52)

$$\frac{1}{\gamma}(x-\varepsilon)^\gamma < u_n(x), \qquad n \geqq N(\varepsilon, \gamma). \tag{56}$$

Remark 3. $u_0(x) = -e^{\gamma x}$, $\gamma < 0$, satisfies (53); hence $u_n(x)$ satisfies (55).

Recall that

$$u_n(x_n) \equiv \frac{U_n(x_n)}{k_1(\gamma)\ldots k_n(\gamma)}, \tag{28}$$

and that $\lim_{\gamma\to\infty} k_j(\gamma) = \infty$ and $\lim_{\gamma\to-\infty} k_j(\gamma) = 0$ [see (26)]. Thus there can be no $(1/\gamma)x^\gamma$, where γ is finite, to which $u_n(x)$ in (55) or (56) converges. What we can say is that the relative risk aversion function $q_n^*(x)$ of $u_n(x)$ tends to $+\infty$ in (55) and to $-\infty$ in (56).

The functions belonging to the class (51) will be taken up in the following sections.

4. The main theorem

We now state our central result.

Theorem 4. *Let $u_0(x_0)$ be a continuous and monotone increasing (terminal wealth utility) function defined for $x_0 \geqq c \geqq 0$ satisfying*[5]

$$\frac{1}{\gamma}(x_0-d)^\gamma - B(x_0-d) \leqq U_0(x_0) \leqq \frac{1}{\gamma}(x_0+d)^\gamma + A(x_0+d), \quad x_0 \geqq c, \tag{57}$$

for some numbers γ and $d \geqq c$, where $U_0(x_0)$ is given by (27) *and $B(x)$, $A(x) \geqq 0$ are defined for $x \geqq 0$. Moreover, let $B(x)$ and $A(x)$ be such that*

(*i*) *when $\gamma > 0$,*

$$B(x) = A(x) = Kx\bar{\gamma}, \tag{58}$$

where $K > 0$, $0 < \bar{\gamma} < \gamma$;

(*ii*) *when $\gamma < 0$,*

$$\frac{1}{\gamma}(x_0+d)^\gamma + A(x_0+d) < 0, \quad x_0 \geqq -d, \tag{59}$$

and

$$\frac{k_n(\gamma, x)}{k_n(\gamma)} \leqq k < 1, \qquad \frac{\bar{p}_n(q, x)}{p_n(q, x)} \leqq k, \quad n = 1, 2, \ldots, \tag{60}$$

for all positive $q \leqq 1$ and $x \geqq 0$, where $k_n(\gamma)$ is given by (21),

$$z_{n\gamma}\left(x_n, \frac{-m}{r_0\ldots r_{n-1}}\right)$$

is the solution to (19),

$$k_n(\gamma, x_n - D_n) \equiv \frac{E[B\{x_{n-1}(z_{n\gamma}(x_n, D_{n-1})) - D_{n-1}\}]}{B(x_n - D_n)}, \tag{61}$$

[5]See footnote 3.

and $p_n(q, x)$ and $\bar{p}_n(q, x)$ are defined by

$$\max_{z_n \in Z_n(x_n, -D_{n-1})} E\left[\frac{1}{\gamma}(x_{n-1}+D_{n-1})^\gamma + qA(x_{n-1}+D_{n-1})\right]$$
$$= p_n(q, x_n+D_n)\frac{1}{\gamma}(x_n+D_n)^\gamma + \bar{p}_n(q, x_n+D_n)qA(x_n+D_n),$$
$$n = 1, 2, \ldots; \tag{62}$$

(iii) when $\gamma = 0$, $k_n(\gamma, x)$ and $\bar{p}_n(q, x)$ (given by (61) and (62)) satisfy

$$k_n(\gamma, x),\ \bar{p}_n(q, x) \leqq k < 1, \quad n = 1, 2, \ldots,$$

for all positive $q \leqq 1$ and $x \geqq 0$. Then the induced (utility of wealth) functions $u_n(x_n)$, $n = 1, 2, \ldots$ [given by (28), (16), and (21)] satisfy

$$\frac{1}{\gamma}(x_n-D_n)^\gamma - B_n(x_n) \leqq u_n(x_n) \leqq \frac{1}{\gamma}(x_n+D_n)^\gamma + A_n(x_n),$$
$$x_n \geqq C_n, \quad n = 1, 2, \ldots, \tag{63}$$

where

$$A_n(x_n),\ B_n(x_n),\ D_n \to 0, \tag{64}$$

i.e.,

$$\frac{1}{\gamma}x^\gamma \leqq \lim u_n(x) \leqq \frac{1}{\gamma}x^\gamma.$$

In addition, whenever $U_0(x_0)$ is strictly concave and returns are stationary, there exists, for every $\varepsilon > 0$ and all x on any finite interval bounded away from zero, a number $N(\varepsilon)$ such that the optimal policy $z_n^*(x)$ satisfies

$$\sum_{i=2}^{M_n} |z_{in}^*(x) - v_{in\gamma}x| < \varepsilon, \quad n \geqq N(\varepsilon), \tag{32}$$

where $v_{j\gamma}$ is the solution to (21), i.e., the optimal policy converges to the optimal policy for the utility function $u(x) = (1/\gamma)x^\gamma$.

Proof. We give the proof for $\gamma > 0$ first. [As before, D_n is given by (31).] By Lemma 6, $k_n(\gamma) > k_n(\bar{\gamma})$ and by Corollary 4 there exists a number $k < 1$ such that

$$\frac{k_n(\bar{\gamma})}{k_n(\gamma)} \leqq k, \quad n = 1, 2, \ldots. \tag{65}$$

For the feasible policy $v_{n\gamma}$ in (21) we obtain

$$\bar{k}_n(\gamma) \equiv E[R_n(v_{n\gamma})^{\bar{\gamma}}] \leqq k_n(\bar{\gamma}) \quad \text{if } \bar{\gamma} > 0. \tag{66}$$

Hence for $\gamma > \bar{\gamma} > 0$

$$k_1(\gamma)\frac{1}{\gamma}(x_1-D_1)^\gamma - \bar{k}_1(\gamma)K(x_1-D_1)^{\bar{\gamma}}$$

$$\leqq \max_{z_1\in Z_1(x_1,\,D_0)} E\left[\frac{1}{\gamma}(x_0-D_0)^\gamma - K(x_0-D_0)^{\bar{\gamma}}\right] \leqq U_1(x_1)$$

$$\leqq \max_{z_1\in Z_1(x_1,\,-D_0)} E\left[\frac{1}{\gamma}(x_0+D_0)^\gamma + K(x_0D_0)^{\bar{\gamma}}\right]$$

$$\leqq k_1(\gamma)\frac{1}{\gamma}(x_1+D_1)^\gamma + k_1(\bar{\gamma})K(x_1+D_1)^{\bar{\gamma}}, \quad x_1 \geqq C_1,$$

again using Lemma 4. By recursion and dividing through by $k_1(\gamma)\ldots k_n(\gamma)$ we obtain, using (28), (65), and (66),

$$\frac{1}{\gamma}(x_n-D_n)^\gamma - k^nK(x_n-D_n)^{\bar{\gamma}} \leqq u_n(x_n)$$

$$\leqq \frac{1}{\gamma}(x_n+D_n)+k^nK(x_n+D_n)^{\bar{\gamma}},$$

$$x_n \geqq C_n, \tag{67}$$

where $k^nK(x\pm D_n)^{\bar{\gamma}} \to 0$ since $k<1$ and $D_n \leqq d$, and $D_n \to 0$ because $\eta_1 > 1$ [see (31)].

When $\gamma < 0$, we obtain from (61) and Lemma 4, since $z_{n\gamma}(x_n, D_{n-1})$ is a feasible policy,

$$\frac{k_1(\gamma)}{\gamma}(x_1-D_1)^\gamma - k_1(\gamma, x_1-D_1)B(x_1-D_1)$$

$$\leqq \max_{z_1\in Z_1(x_1,\,D_0)} E\left[\frac{1}{\gamma}(x_0-D_0)^\gamma - B(x_0-D_0)\right] \leqq U_1(x_1). \tag{68}$$

(59), (62), and Lemma 4 give

$$U_1(x_1) \leqq \max_{z_1\in Z_1(x_1,-D_0)} E\left[\frac{1}{\gamma}(x_0+D_0)^\gamma + A(x_0+D_0)\right]$$

$$= \frac{p_1(1, x_1+D_1)}{\gamma}(x_1+D_1)^\gamma + \bar{p}_1(1, x_1+D_1)A(x_1+D_1)$$

$$< 0, \qquad x_1 \geqq C_1, \tag{69}$$

where, by (21)

$$p_1(1, x_1+D_1) \geqq k_1(\gamma).$$

Defining $c_1(x_1)$ by

$$c_1(x_1)p_1(1, x_1+D_1) = k_1(\gamma), \quad x_1 \geqq -D_1, \tag{70}$$

we have

$$0 < c_1(x_1) \leqq 1, \tag{71}$$

and hence from (69) and (70)

$$U_1(x_1) \leqq \frac{k_1(\gamma)}{\gamma}(x_1+D_1)^\gamma + c_1(x_1)\bar{p}_1(1, x_1+D_1)A(x_1+D_1)$$
$$< 0, \quad x_1 \geqq C_1. \tag{72}$$

By assumption (60)

$$\frac{k_n(\gamma, x_n-D_n)}{k_n(\gamma)} \leqq k, \qquad x_n \geqq D_n, \quad n = 1, 2, \ldots, \tag{73}$$

$$\frac{\bar{p}_n(k^{n-1}, x_n+D_n)}{p_n(k^{n-1}, x_n+D_n)} \leqq k, \quad x_n \geqq -D_n, \quad n = 1, 2, \ldots, \tag{74}$$

for some $0 < k < 1$. Dividing (68) and (72) by $k_1(\gamma)$, we obtain, using (28), (70), (73), and (74),

$$\frac{1}{\gamma}(x_1-D_1)^\gamma - kB(x_1-D_1) \leqq u_1(x_1)$$
$$\leqq \frac{1}{\gamma}(x_1+D_1)^\gamma + kA(x_1+D_1),$$
$$x_1 \geqq C_1. \tag{75}$$

Since $k < 1$ in (75) and $D_0 > D_1 > D_2 \ldots$, we can repeat the process for $n = 2, 3, \ldots$, which gives

$$\frac{1}{\gamma}(x_n-D_n)^\gamma - k^n B(x_n-D_n) \leqq u_n(x_n),$$
$$\leqq \frac{1}{\gamma}(x_n+D_n)^\gamma + k^n A(x_n+D_n),$$
$$x_n \geqq C_n. \tag{76}$$

Again, both bounds converge to $(1/\gamma)x^\gamma$, which completes the proof of (63) and (64) for $\gamma < 0$; the proof for $\gamma = 0$ is similar. The proof of (32) [when $u_0(x)$ is strictly concave and returns are stationary] is essentially the same as that used in Theorem 2 and is therefore omitted.

The following corollary is immediate:

Corollary 7. *Assume that* $u_0(x_0)$ *satisfies* (57) *for some* $\gamma > 0$, *where* $A(x) = B(x) \geqq 0$ *for* $x \geqq 0$, *and*

$$k\, k_n(\gamma) \geqq \frac{\max\limits_{z_n \in Z_n(x_n,\, 0)} E[A(x_{n-1})]}{A(x_n)}, \quad n = 1, 2, \ldots,$$

for all $x_n \geqq 0$ *and some* $k < 1$. *Then the induced utility functions* $u_n(x_n)$, $n = 1, 2, \ldots$, *satisfy* (76) *and hence* (63) *and* (64); *and if* $\gamma < 1$ *the optimal policy* $z_n^*(x)$ *satisfies* (32) *under stationary returns.*

Remark 4. Let $u_0(x_0) \equiv \sum_{i=1}^{I} (a_i/\gamma_i)x_0^{\gamma_i}$, where the a_i are positive constants and $\max_i \gamma_i > 0$. Then $U_0(x_0)$ satisfies (57) and (58) and hence $u_n(x_n)$, $n = 1, 2, \ldots$, satisfies (63) and (64) for $\gamma = \max_i \gamma_i$, and if $\gamma < 1$ (and returns are stationary) the optimal policy $z_n^*(x)$ satisfies (32).

In words, if the terminal utility of wealth function is a positive linear combination of power functions and at least one power is positive, the induced utility of wealth functions converge to that power function in the linear combination which has the largest exponent.

Remark 5. Let $u_0(x_0) = \sum_{i=1}^{I} (a_i/\gamma_i)x_0^{\gamma_i}$, where the a_i are positive constants, $\gamma \equiv \max_i \gamma_i \leqq 0$, and

$$\frac{k_n(\gamma_i;\gamma)}{k_n(\gamma)} \leqq k < 1, \quad \text{all } \gamma_i \neq \gamma,\ n = 1, 2, \ldots, \tag{77}$$

where $k_n(\gamma_i;\gamma) \equiv E[R_n(v_{n\gamma})^{\gamma_i}]$. Then $U_0(x_0)$ satisfies the conditions of Theorem 4, i.e., $u_n(x)$ converges to $(1/\gamma)x^\gamma$ and if returns are stationary $z_n^*(x)$ satisfies (32).

A sufficient, but not necessary, condition for (77) to hold is

$$\Pr\{R_n(v_{n\gamma}) \geqq 1\} = 1, \quad n = 1, 2, \ldots, \tag{78}$$

as is easily verified; (78) is consistent with assumptions (2) and (3) and may well be satisfied for sufficiently risk-averse investors.

Significant extension of the boundaries $A(x)$ and $B(x)$ beyond the limits given in (58) is not possible for general return distributions since if we attempt to make $\bar{\gamma} = \gamma$, convergence of $u_n(x)$ to $(1/\gamma)x^\gamma$ need not occur, as the following counter-example shows.

Let $u_0(x_0)$ be strictly concave and such that

$$\frac{1}{\gamma}x^\gamma \leqq U_0(x) \leqq \frac{s}{\gamma}x^\gamma, \quad s > 1, \tag{79}$$

where $U_0(x)$ has contact with the lower bound at points $\ldots \frac{1}{4}x', x', 4x', 16x' \ldots$ and the upper bound at points $\ldots \frac{1}{4}x'', x'', 4x'', 16x'' \ldots$. Assume that $R_j(v_{j\gamma})$ [where $v_{j\gamma}$, as before, is the policy which maximizes (21)] assumes the values $\frac{1}{2}$ and 2 with equal probability in each period and let $v_n^*(x_n)x_n$ be the optimal policy with n periods to go. Then it is readily verified [recall that $U_0(x_0)$ is strictly concave] that $v_1^*(x_1) = v_{1\gamma}$ for $x_1 = \ldots \frac{1}{2}x', 2x', 8x', \ldots \frac{1}{2}x'', 2x'', 8x'', \ldots$ which gives

$$U_1(x_1) = \frac{k(\gamma)}{\gamma} x_1^\gamma, \qquad x_1 = \ldots \tfrac{1}{2}x', 2x', 8x', \ldots,$$

$$U_1(x_1) = \frac{sk(\gamma)}{\gamma} x_1^\gamma, \qquad x_1 = \ldots \tfrac{1}{2}x'', 2x'', 8x'', \ldots.$$

Continuing in the same fashion, each of the functions $u_2(x_2)$, $u_3(x_3)$, ... is continuous and strictly concave by Lemma 3 and satisfies

$$\frac{1}{\gamma} x^\gamma \leqq u_n(x) \leqq \frac{s}{\gamma} x^\gamma, \tag{80}$$

by Lemma 4, with $u_n(x)$ alternatingly touching both boundaries at points $\ldots \frac{1}{4}x', x', 4x', \ldots \frac{1}{4}x'', x'', 4x'', \ldots$ (n even) or points $\ldots \frac{1}{2}x', 2x', 8x', \ldots \frac{1}{2}x'', 2x'', 8x'', \ldots$ (n odd). Thus, $u_n(x)$ does not converge to an isoelastic function of any power (or any other function).

5. A sufficient condition for convergence

We have noted that (63) and (64) fail to hold if the terminal utility function $U_0(x_0)$ satisfies (52) or (53) or oscillates rather violently around some $(1/\gamma)x^\gamma$ and does so indefinitely. However, failure of the *bounds* in (57) to converge does not necessarily mean that $u_n(x_n)$ fails to converge to an isoelastic function. For example, if $R_j(v_j)$ were to have a uni-modal density, the function in (79) would converge in the sense that s, instead of remaining fixed, could be reduced to 1 in (80).

The theorem below illustrates the kind of requirement that must be satisfied for a utility function not satisfying (57) to converge to an isoelastic one. The following notation and lemma will be needed. Let

$$u_n(x_n) \equiv \frac{U_n(x_n)}{k_{21} \ldots k_{2n}}, \qquad n = 1, 2, \ldots, \tag{81}$$

where $U_1(x_1)$, $U_2(x_2)$, ... are given by (16) and $k_{21}, k_{22}, \ldots$ are given by

$$k_{in} \equiv \gamma_{i,n-1} \max_{v_n \in V_n} E\left[\frac{1}{\gamma_{i,n-1}} R_n(v_n)^{\gamma_{i,n-1}}\right], \qquad i = 1, 2;\ n = 1, 2, \ldots; \tag{82}$$

in (82), γ_{2n} is the smallest value of γ such that

$$U_n(x_n) \leqq k_{21} \ldots k_{2n} (x_n + D_{2n})^\gamma, \qquad x_n \geqq C_n,\ n = 1, 2, \ldots. \tag{83}$$

Also, let

$$k_n(x) \equiv \frac{U_n(x)}{U_{n-1}(x)}, \quad n = 1, 2, \ldots. \tag{84}$$

Lemma 7. *If*

$$\frac{A(x) - x^{\gamma_1}}{x^{\gamma_2} - x^{\gamma_1}} \geqq \varepsilon,$$

where $\gamma_1 < \gamma_2$, $x > 1$, $0 < \varepsilon < 1$, *then*

$$A(x) > x^{\gamma_1 + \varepsilon(\gamma_2 - \gamma_1)}, \quad x > 1.$$

The proof is based on the strict convexity of x^γ in γ, $x > 1$.

As an example, for utility functions unbounded above we can state:

Theorem 5. *Let* $U_0(x_0)$ *be a continuous and monotone increasing (terminal wealth utility) function defined for* $x_0 \geqq c \geqq 0$ *satisfying*[6]

$$(x_0 - D_{10})^{\gamma_{10}} \leqq U_0(x_0) \leqq (x_0 + D_{20})^{\gamma_{20}}, \quad x_0 \geqq c, \tag{85}$$

for some numbers $\gamma_{20} > \gamma_{10} \geqq 0$ *and* $D_{10} > c$, $D_{20} \geqq 0$. *Then the induced utility functions* $u_n(x_n)$, $n = 1, 2, \ldots$, *[given in* (81), (16), *and* (82)] *satisfy*

$$(x_n - D_{1n})^{\gamma_{1n}} \leqq u_n(x_n) \leqq (x_n + D_{2n})^{\gamma_{2n}}, \tag{86}$$

where both bounds are tight,

$$D_{in} \equiv \frac{D_{i0}}{r_1 \ldots r_n}, \quad i = 1, 2, \quad n = 1, 2, \ldots, \tag{87}$$

$\gamma_{21} \geqq \gamma_{22} \geqq \ldots$, and

$$\gamma_{2n} - \gamma_{1n} \leqq (1-\varepsilon)^{n-L}(\gamma_{20} - \gamma_{10}), \quad n \geqq L \geqq N, \tag{88}$$

for some number L whenever there exists an ε $(0 < \varepsilon < 1)$ *such that*

$$\frac{[k_n(x_n)/k_{2n}]u_{n-1}(x_n) - (x_n - D_{1n})^{\gamma_{1,n-1}}}{(x_n + D_{2n})^{\gamma_{2,n-1}} - (x_n - D_{1n})^{\gamma_{1,n-1}}} \geqq \varepsilon, \quad x_n \geqq D_{1n}, n \geqq N. \tag{89}$$

Proof. By (81) and (83) we have

$$u_n(x_n) \leqq (x_n + D_{2n})^{\gamma_{2n}}, \quad x_n \geqq C_n, n = 1, 2, \ldots, \tag{90}$$

where $\gamma_{21} \geqq \gamma_{22} \geqq \ldots$ by Lemma 4. For $n = N-1$ there also exists, since we may choose D_{10} arbitrarily large, a $\gamma_{1,N-1}$ (which may be less than γ_{10}) such that

$$u_{N-1}(x_{N-1}) \geqq (x_{N-1} - D_{1,N-1})^{\gamma_{1,N-1}}, \quad x_{N-1} \geqq D_{1,N-1}, \tag{91}$$

[6]See footnote 3.

with equality holding for some values of x_{N-1}. Using (81) and (84) we obtain

$$u_n(x_n) = \frac{k_n(x_n)}{k_{2n}} u_{n-1}(x_n), \quad n = 1, 2, \ldots .$$

Thus, by assumption (89),

$$\frac{u_n(x_n) - (x_n - D_{1n})^{\gamma_{1,n-1}}}{(x_n + D_{2n})^{\gamma_{2,n-1}} - (x_n - D_{1n})^{\gamma_{,n-1}}} \geqq \varepsilon, \quad x_n \geqq D_{1n}, n \geqq N. \tag{92}$$

By Lemma 7 (remembering that $x_n + D_{2n} > x_n - D_{1n}$), (92) gives

$$u_N(x_N) > (x_N - D_{1N})^{\gamma_{1,N-1} + \varepsilon(\gamma_{2,N-1} - \gamma_{1,N-1})}, \quad x_N > D_{1N} + 1.$$

But since $\gamma_{1,N-1} \geqq 0$,

$$(x_N - D_{1N})^{\gamma_{1,N-1} + \varepsilon(\gamma_{2,N-1} - \gamma_{1,N-1})} \leqq (x_N - D_{1N})^{\gamma_{1,N-1}}, \quad D_{1N} \leqq x_N \leqq D_{1N} + 1,$$

so that

$$u_N(x_N) \geqq (x_N - D_{1N})^{\gamma_{1N}}, \quad x_N \geqq D_{1N},$$

where, with equality holding for some x_{1N},

$$\gamma_{1N} > \gamma_{1,N-1} + \varepsilon(\gamma_{2,N-1} - \gamma_{1,N-1}).$$

Setting $\gamma_{2,N-1} - \gamma_{1,N-1} = K(\gamma_{20} - \gamma_{10})$, we obtain

$$\gamma_{2N} - \gamma_{1N} \leqq (1 - \varepsilon) K(\gamma_{20} - \gamma_{10}).$$

But (89) holds for all $n \geqq N$ so that by recursion

$$\gamma_{2n} - \gamma_{1n} \leqq (1 - \varepsilon)^{n-N+1} K(\gamma_{20} - \gamma_{10}), \quad n \geqq N,$$
$$\rightarrow 0,$$

which concludes the proof.

In fig. 1, the ratio (89) is depicted as $A/(A+B)$. The requirement that it exceed ε seems rather innocuous. Recall also that it need not be operative until $n \geqq N$. When equality holds in (85) [which is only possible over finite intervals of large x if (57) and (58) are violated since $D_{10} > c$], $A/(A+B)$ may in fact be zero for some x when n is small. But (3) and Lemma 5 insure that after a finite number of iterations, contact between $U_n(x_n)$ and the lower bound is limited to single points; and the same properties assure that from that point on the ratio will be non-zero.

6. Discussion

The reader will undoubtedly have noted that the preceding results do not depend on the terminal utility function being monotone everywhere. While

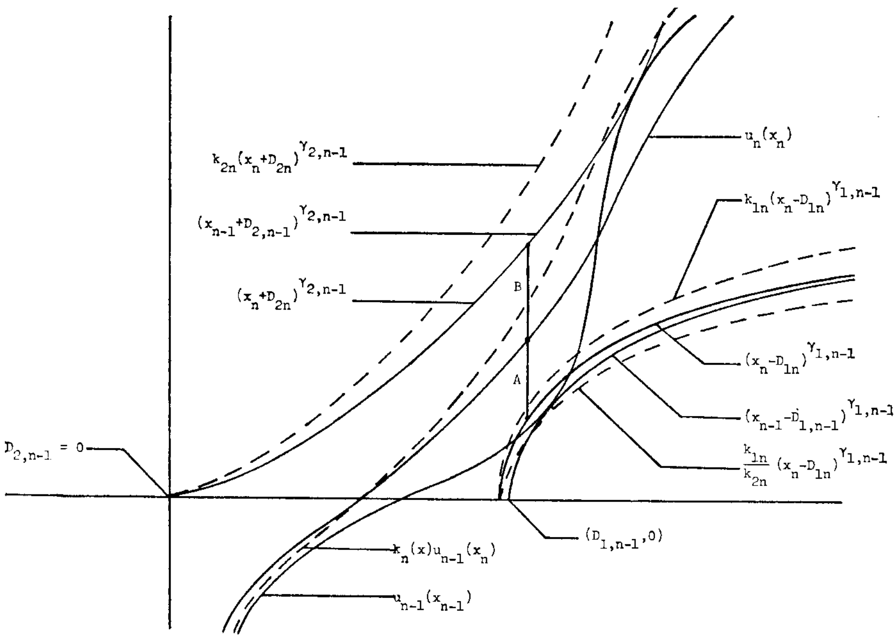

Fig. 1. Abscissa: wealth, ordinate: utility.

many locally decreasing functions satisfy the necessary bounds, they were ruled out on *a priori* grounds. The same observation holds regarding the presence of the solvency constraint (4) and the concomitant assumption that $c \geqq 0$; mathematically everything goes through for $c < 0$.

If the analysis is restricted to functions $U_0(x_0)$ which are risk averse for large x_0, the upper bound in (1) can be relaxed to the point where only the first moments are bounded.

The preceding results can also be extended, with some complication, to the case in which there is no riskless asset, provided the returns on the risky assets are sufficiently favorable.

Since utility functions of type (52) are clearly of no interest, functions of type (53) seem to be of at most limited interest, and (89) would, for realistic return structures, appear satisfied by many functions not satisfying the requirements of Theorem 4, the asymptotic relevance of the class (17) extends to a substantial portion of the terminal utility functions one might expect to encounter in real situations. The significance of this is threefold. First, 'qualifying' investors can without sacrifice behave myopically when their horizon is distant (even though most of them can only do so at a considerable price when the horizon is near), since (17) is the only class for which optimal behavior is always myopic. Second, since (17) exhibits the separation property but each γ yields a different optimal mix of risky assets (which is not a linear combination of other mixes) [Cass and

Stiglitz (1970), Hakansson (1970a)], one mutual fund for each γ associated with the set of terminal utility functions is both necessary and, in the absence of differing return assessments, sufficient to serve all such long-run 're-investors' in the economy. Finally, all long-run investors associated with a $\gamma \leqq \gamma^*$, where $\gamma^* > 0$ (and depends on $F_1, \ldots, F_n, r_1, \ldots, r_n$), will invest in such a way that $\Pr\{x_j < x_1\} \to 0$, i.e., they will not risk ruin [Hakansson and Miller (1973)].

Writing on a related topic, Goldman (1974) showed that the 'growth-optimal' or logarithmic policy is a poor one to pursue for investors with bounded utility functions. That result can be said to be confirmed in the present paper in the sense that when the optimal policy for a bounded function does converge to an isoelastic policy, it is never to the logarithmic one ($\gamma = 0$) but to one with a negative power, i.e., to a more conservative policy (see also Corollary 5). In fact, it is clear from the present results that the class of utility functions for which the logarithmic policy is asymptotically optimal is small indeed – even though that policy has the noteworthy property that it almost surely leads to more *capital* in the long run than any other (significantly different) policy.

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