AN INDUCED THEORY OF THE FIRM UNDER RISK: THE PURE MUTUAL FUND

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In two previous articles [11] and [12] a family of normative models of the individual's economic decision problem under risk was presented. At the same time, certain implications of these models with respect to individual behavior were deduced for a class of utility functions. This paper will show that these models also give rise to an induced theory of the formation and operation of firms under risk for the same class of utility functions.

Section I considers the interrelationship between investment and consumption decisions and discusses the basic approach of the present study to development of a normative theory of the firm. A subset of the decision models developed in the earlier papers is then reconstructed in Sections II and III; the solutions for this subset and a review of the properties of the optimal strategies are given in Section IV.

On the basis of the foregoing models of the individual's economic decision problem, it is shown in Section V that, starting with a collection of heterogeneous individuals, each of whom is bent on maximizing (his own) utility from consumption over time, there exists a basis for the formation of firms by *subcollections* of individuals. Each of these subcollections, in turn, possesses heterogeneity with respect to age, wealth, noncapital income, and impatience to consume. Each firm so formed is also found, in Section VI, to have a well-defined (unique) objective function, which may be interpreted as imputing a precise meaning to the term "profit maximization" under risk and with respect to time. Since the capital structure of the firm turns out to be unimportant, an unexpected tie-in with Proposition I of Modigliani and Miller [16, p. 268] is also obtained.

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The question of stock issues, capitalization, and dividends is considered in Section VII. In the case of limited liability on the part of the firm, it is shown in Section VIII that there is an upper limit on the firm's feasible debt-equity ratio. In Section IX, it is found that a subset of the firms examined in the current paper has as its objective the maximization of the expected growth rate of capital.

Some of the applications to which the model lends itself are discussed in Sections X and XI. It is noted that the model is particularly applicable to the balanced mutual fund as well as to endowed educational and charitable organizations. Finally, Section XII indicates how the theory extends to a more general case in that the owners of a firm of the preceding type may differ not only in age, wealth, noncapital income, and impatience to consume, but also in survival probabilities, "strength" of bequest motive, and insurance coverage.

I. The Firm, the Individual, and Consumption

Any normative model presumes the existence and availability of an objective function. Thus, the derivation of "optimal" investment strategies, for example, is contingent upon proper and precise specification of a maximand. In the case of the firm, there is wide disagreement as to what its objective should be, a disagreement which shows no sign of narrowing. Even if one were to adopt the classical postulate of profit maximization, one immediately runs into conceptual difficulties: what does it mean to maximize profits under risk or uncertainty? And even in the case of certainty one faces the intertemporal question: *when* do we maximize profits?

Since all claims to the capital of the firm reside in individuals, it seems reasonable that the objective of the firm should be at least *grounded* in the objectives of the individual investors of capital. If, therefore, one views the objective of the firm as *derived*, in some fashion, from the investment objectives of individuals, the latter become a sensible starting point in examination of investment objectives in general.

Turning, therefore, to the individual, we find that while the object of investment activity is capital, capital *per se* offers nothing to him until it is spent. Thus, the value, or utility, of capital is determined by the enjoyment derived from the consumption it buys.[1] Since consumption, therefore, is the

[1]Broadly defined, consumption would include such things as bequests and money allocated to gambling for pleasure.

ultimate source of all pecuniary utility, we are ultimately led in our search for optimal investment strategies to consider individuals' utility functions of alternative consumption programs. But, this is exactly what we would do if we were interested in determining his optimal consumption strategies. It is clear at this juncture that the individual's economic choice problem is twofold: *how much* to invest (or alternatively, how much of one's capital to consume presently), and *how* to invest that capital which is not presently consumed; i.e., how to allocate it among available opportunities. This problem, which may be called the individual's economic decision problem, was examined in [11] and [12]. In this paper, we find that models developed in earlier papers to study the individual and the implications obtained there also form the basis for a theory of the firm.

II. Assumptions and Notation

We begin by assuming that we have an economy consisting of individuals and nonprofit organizations in which opportunities for decision occur at discrete, equally spaced points in time called decision points. The objective of each individual is postulated to be the maximization of expected utility from consumption over time where the horizon is arbitrarily distant. The individual's resources are assumed to consist of an initial capital position (which may be negative) and a noncapital income stream, which is known with certainty but may possess any time-shape. The individual faces both financial opportunities (borrowing and lending) and an arbitrary number of productive investment opportunities. The interest rate is presumed to be known but may have any time-shape; returns from the productive opportunities are assumed to be random variables whose probability distributions may differ from period to period but are assumed to satisfy the "no-easy-money-condition." Although no limit is placed on borrowing, it is postulated that all debt must be fully secured at all times.

The following notation and further assumptions will be employed for a given individual:

(a) c_j = amount of consumption in period j, where $c_j \geq 0$ (decision variable),

(b) $U_j(c_j, c_{j+1}, \ldots, c_J)$ = the utility function, defined on all possible consumption programs $(c_j, c_{j+1}, \ldots, c_J)$, $j = 1, \ldots, J$. The class of functions to be considered is that of the form

(1) $U_j(c_j, c_{j+1}, \ldots, c_J) = u(c_j) + \alpha u(c_{j+1}) + \ldots + \alpha^{J-j} u(c_J) \quad 0 < \alpha < 1.$

u(c) is assumed to be monotone increasing, twice differentiable, and strictly concave for $c \geq 0$. The objective at decision point j is to maximize $E[U_j(c_j, \ldots, c_J)]$, i.e., the expected utility derived from consumption over future time.[2] The term α is the patience factor reflecting the individual's time preference.

(c) x_j = amount of capital (debt) on hand at decision point j (the beginning of the j^{th} period) (state variable),

(d) y_j = income received from noncapital sources at the *end* of period j, where $0 \leq y_j < \infty$,

(e) M_j = the number of investment opportunities available in period j,

(f) S_j = the subset of investment opportunities which can be sold short[3] in period j,

(g) z_{1j} = amount lent in period j (negative z_{1j} indicate borrowing) (decision variable),

(h) z_{ij} = amount invested in opportunity i, $i = 2, \ldots, M_j$, at the beginning of the j^{th} period (decision variable),

(i) $r_j - 1$ = rate of interest in period j, where $r_j > 1$,

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[2]While we make use of the expected utility theorem, we assume that the von Neumann-Morgenstern postulates [21] have been modified in such a way to permit unbounded utility functions.

[3]We shall arbitrarily define a short sale as the opposite of a long investment; i.e., if the individual sells opportunity i short in the amount θ, he will receive θ immediately (to do with as he pleases) in return for the obligation to pay the transformed value of θ at the end of the period.

(j) β_{ij} = transformation of capital invested in opportunity i, where $i = 2,\ldots,M_j$, in the j^{th} period, per unit of capital so invested (random variable); i.e., if we invest an amount θ in i at the beginning of the period, we will obtain $\beta_{ij}\theta$ at the end of that period (stochastically constant returns to scale, no transaction costs or taxes, perfect liquidity).

(k) F_j = joint distribution function of the β_{ij}, $i = 2,\ldots,M_j$. The $\{F_j\}$ are assumed to be known and independent for all j, and to have the following additional properties.

(2) $$0 \leq \beta_{ij} < \infty \quad \text{for all } i, j$$

(3) $$\Pr\{\sum_{i=2}^{M_j} (\beta_{ij}-r_j)\theta_i < 0\} > 0$$

for all j and all finite θ_i such that $\theta_i \geq 0$ for all $i \notin S_j$ and $\theta_i \neq 0$ for at least one i.

(l) $f_j(x_j)$ = expected utility obtainable from consumption over all future time, evaluated at decision point j, when initial capital is x_j and an optimal strategy is followed with respect to consumption and investment,

(m) Y_j = present value at decision point j of the noncapital income stream capitalized at the (borrowing) rate of interest, i.e.,

$$Y_j \equiv \frac{y_j}{r_j} + \frac{y_{j+1}}{r_j r_{j+1}} + \ldots + \frac{y_J}{r_j \ldots r_J}$$

(n) $c_j^*(x_j)$ = an optimal consumption strategy at decision point j,

(o) $z^*_{1j}(x_j)$ = an optimal lending strategy at decision point j,

(p) $z^*_{ij}(x_j)$ = an optimal investment strategy for opportunity i, $i = 2,\ldots,M_j$, at decision point j.

The limitations of utility functions of the form (1) are well known and need not be elaborated here. Property (3) will be referred to as the "no-easy-money condition." In essence, this condition states:

- that no combination of productive investment opportunities exists which provides, with probability 1, a return at least as high as the (borrowing) rate of interest;
- that no combination of short sales exists in which the probability is zero that a loss will exceed the (lending) rate of interest;
- that no combination of productive investments made from the proceeds of any short sale can guarantee against loss.

For these reasons, (3) may be viewed as a condition that must be satisfied in equilibrium by the prices of the various assets in the market.

Consumption and investment decisions are assumed to be made at the beginning of each period. The amount allocated to consumption is assumed to be spent immediately or, if spent gradually over the period, to be set aside in a nonearning account. We also assume that any debt incurred by the individual must at all times be fully secured, i.e., that the individual must be solvent at each decision point. In view of the "no-easy-money condition" (3), this implies that his (net) debt cannot exceed the present value, on the basis of the (borrowing) rate of interest, of his noncapital income stream at the end of any period.

III. Derivation of the Basic Model

We shall now identify the relationship which determines the amount of capital (debt) on hand at each decision point in terms of the amount on hand at the previous decision point. This leads to the difference equation

$$(4) \qquad x_{j+1} = r_j z_{1j} + \sum_{i=2}^{M_j} \beta_{ij} z_{ij} + y_j \qquad j = 1,2,\ldots,J$$

where

$$(5)\qquad \sum_{i=1}^{M_j} z_{ij} = x_j - c_j \qquad j = 1,2,\ldots,J.$$

The first term of (4) represents the payment of the debt or the proceeds from savings; the second term, the proceeds from productive investments; and the third term, the noncapital income received. Combining (4) and (5) we obtain

$$(6)\qquad x_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij}-r_j)z_{ij} + r_j(x_j-c_j) + y_j \qquad j = 1,2,\ldots,J.$$

This is the difference equation, then, which governs the process we are studying.

The definition of $f_j(x_j)$ may formally be written

$$(7)\qquad f_j(x_j) \equiv \text{Max } E[U_j(c_j,c_{j+1},\ldots,c_J)]|x_j \qquad j = 1,\ldots,J.$$

From (1) we obtain, by the principle of optimality,[4] for all j,

$$(8)\qquad f_j(x_j) = \text{Max } E[u(c_j) + \alpha\{\text{Max } E[U_{j+1}(c_{j+1},\ldots,c_J)]|x_{j+1}\}]|x_j$$

since we have assumed the $\{\beta_{ij}\}$ to be independently distributed with respect to time j. By (7), (8) reduces to

$$(9)\qquad f_j(x_j) = \text{Max}\{u(c_j) + \alpha E[f_{j+1}(x_{j+1})]\} \qquad \text{all } j.$$

Using (6), (9) becomes

P1:

$$(10)\quad f_j(x_j) = \underset{c_j,\{z_{ij}\}}{\text{Max}} \{u(c_j) + \alpha E[f_{j+1}(\sum_{i=2}^{M_j}(\beta_{ij}-r_j)z_{ij} + r_j(x_j-c_j) + y_j)]\}$$

$$j = 1,\ldots,J$$

where

$$(11)\qquad f_{J+1}(x_{J+1}) \equiv 0 \qquad \text{for finite } J$$

subject to

$$(12)\qquad c_j \geq 0 \qquad j = 1,\ldots,J,$$

$$(13)\qquad z_{ij} \geq 0 \quad i \notin S_j \qquad j = 1,\ldots,J,$$

[4]The principle of optimality states that an optimal strategy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision [2, p. 83].

and

$$(14) \qquad Pr\{\sum_{i=2}^{M_j}(\beta_{ij}-r_j)z_{ij} + r_j(x_j-c_j) + y_j \geq -Y_{j+1}\} = 1 \qquad j = 1,\ldots,J$$

at each decision point. Equation (14), of course, represents the solvency constraint. In the stationary, infinite horizon, case P1 reduces to
P2:

$$(10a) \qquad f(x) = \max_{c,\{z_i\}} \{u(c) + \alpha E[f(\sum_{i=2}^{M}(\beta_i-r)z_i + r(x-c) + y)]\}$$

subject to

$$(12a) \qquad c \geq 0,$$

$$(13a) \qquad z_i \geq 0 \qquad i \notin S,$$

and

$$(14a) \qquad Pr\{\sum_{i=2}^{M}(\beta_i-r)z_i + r(x-c) + y \geq -Y\} = 1$$

at *each* decision point. P1 and P2 represent a generalization of Phelps' model of personal saving [17] and may be viewed as a formalization of the models of the individual developed in [20] and [8] under risk.

Since x represents capital, $f_j(x)$ is clearly the utility of capital at the j^{th} decision point. Instead of being assumed, as is generally the case, the utility function of money has in this model been induced from inputs which are more basic than the preferences for money itself; it clearly will not be known until P1 has been solved. As (10) shows, $f_j(x)$ depends in general on the individual's preferences with respect to consumption, his noncapital income stream, his age, future interest rates, and the available investment opportunities and their riskiness. Are not these the very factors that an individual, given the task of constructing his utility of money, would consider? Since money is only a means to an end, it should therefore come as no surprise that its utility is dependent on the utility of the end and the opportunity for achieving it.

IV. Summary of Previous Results

In general, a solution to P1 and P2 appears accessible only by numerical methods. However, in [11] a solution was obtained in closed form for the utility functions

(15) $$u(c) = c^{\gamma} \qquad 0 < \gamma < 1 \quad \text{Model I,}$$

(16) $$u(c) = -c^{-\gamma} \qquad \gamma > 0 \quad \text{Model II,}$$

(17) $$u(c) = \log c \qquad \text{Model III,}$$

and

(18) $$u(c) = -e^{-\gamma c} \qquad \gamma > 0 \quad \text{Model IV.}$$

In [18], it was noted that (15)-(17) are the only monotone increasing and strictly concave utility functions for which the proportional risk aversion index

(19) $$q^{*}(c) \equiv -\frac{u''(c)c}{u'(c)}$$

is a positive constant, and that (18) is the only monotone increasing and strictly concave utility function for which the absolute risk aversion index

(20) $$q(c) \equiv -\frac{u''(c)}{u'(c)}$$

is a positive constant.

For the class of one-period utility functions (15)-(18), the solution to P1 and P2 was found to be given by the following theorems:

<u>THEOREM 1</u>: Let α, $\{\beta_{ij}\}$, r_j, M_j, S_j, F_j, y_j, and Y_j be defined as in Section II. Moreover, let $u(c)$ be one of the functions (15), (16), or (17). Then, a solution to P1 and P2 exists for $x_j \geq -Y_j$ and is given, for all j, by

(21) $$f_j(x_j) = A_j u(x_j+Y_j) + C_j,$$

(22) $$c_j^{*}(x_j) = B_j(x_j+Y_j),$$

(23) $$z_{1j}^{*}(x_j) = (1-B_j)(1-v_j^{*})(x_j+Y_j) - Y_j,$$

and

(24) $$z_{ij}^{*}(x_j) = (1-B_j)v_{ij}^{*}(x_j+Y_j) \qquad i = 2,\ldots,M_j$$

where $A_j > 0$, $B_j > 0$, and C_j are constants[5] which depend on α, γ, and

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[5]For example, in Model II, these constants have the values

(a) $$A_j = \left\{1 + [\alpha(-k_j)]^{\frac{1}{\gamma+1}} + \ldots + [\alpha(-k_j)\alpha(-k_{j+1})\ldots\alpha(-k_{J-1})]^{\frac{1}{\gamma+1}}\right\}^{1+\gamma},$$

(b) $B_j = A_j^{\frac{-1}{1+\gamma}}$, and

$k_j,\ldots,k_J$, and the constants v^*_{ij} ($v^*_j \equiv \sum_{i=2}^{M_j} v^*_{ij}$) and k_n, $n = j,\ldots,J$, are given by

$$(25) \qquad k_n \equiv E[u(\sum_{i=2}^{M_n}(\beta_{in}-r_n)v^*_{in} + r_n)] = \underset{\{v_{in}\}}{\text{Max}}\ E[u(\sum_{i=2}^{M_n}(\beta_{in}-r_n)v_{in} + r_n)]$$

subject to

$$(26) \qquad v_{in} \geq 0 \qquad\qquad i \notin S_n$$

and

$$(27) \qquad \Pr\{\sum_{i=2}^{M_n}(\beta_{in}-r_n)v_{in} + r_n \geq 0\} = 1$$

provided that $\alpha k_m < 1$ for all $m \geq I$, where I is a positive integer, when the horizon is infinite in the case of Model I. Furthermore, the solution is unique.

<u>THEOREM 2</u>: Let α, $\{\beta_{ij}\}$, r_j, M_j, S_j, F_j, y_j, and Y_j be defined as in Section III. Moreover, let $u(c)$ be the function (18) for $c \geq 0$. Then, a solution to P1 and P2 exists for $x_j \geq -Y_j + x^o_j$ and is given, for all j, by

$$(28) \qquad f_j(x_j) = A_j u[C_j(x_j + Y_j)],$$

$$(29) \qquad c^*_j(x_j) = B_j(x_j + Y_j) + D_j,$$

$$(30) \qquad z^*_{ij}(x_j) = E_j v^*_{ij} \qquad\qquad i = 2,\ldots,M_j,$$

$$(31) \qquad z^*_{1j}(x_j) = x_j - c^*_j(x_j) - \sum_{i=2}^{M_j} z^*_{ij},$$

where $x^o_j > 0$, $A_j > 0$, $B_j > 0$, $C_j > 0$, D_j, and $E_j > 0$ are constants which depend on α, γ, $r_j,\ldots,r_J$, and $k_j,\ldots,k_J$, and the constants

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(c) $C_j = 0$,

while in Model III they are given, in the stationary infinite horizon case, by

(a) $A_j = \dfrac{1}{1-\alpha}$,

(b) $B_j = 1 - \alpha$, and

(c) $C_j = \dfrac{1}{1-\alpha}\log(1-\alpha) + \dfrac{\alpha \log \alpha}{(1-\alpha)^2} + \dfrac{\alpha k}{(1-\alpha)^2}$

for all j.

$v_{ij}^*(v_j^* \equiv \sum_{i=2}^{M_j} v_{ij}^*)$ and k_n, $n = j,\ldots,J$, are given by

$$(32) \qquad k_n \equiv E[-e^{-\sum_{i=2}^{M_n}(\beta_{in}-r_n)v_{in}^*}] = \underset{\{v_{in}\}}{\text{Max}}\ E[-e^{-\sum_{i=2}^{M_n}(\beta_{in}-r_n)v_{in}}]$$

subject to (26) provided that a certain condition holds. In the stationary, infinite horizon case, this condition is given by

$$(33) \qquad \log(-\alpha kr) + b(\bar{v}^*) \geq 0$$

where $b(\bar{v}^*)$ is the greatest lower bound on b such that

$$\Pr\{\sum_{i=2}^{M}(\beta_i - r)v_i^* < b\} > 0$$

and $\bar{v}^* \equiv (v_2^*,\ldots,v_M^*)$. Moreover, the solution is unique.

Properties of the Optimal Strategies

In order to facilitate interpretation of these results, we shall briefly summarize the properties of the optimal strategies (22), (23), (24), (29), (30), and (31). A more complete description is given in [11].

In each of the four models, we note that the optimal consumption function $c_j^*(x_j)$ is linear increasing in capital x_j and in noncapital income Y_j. Whenever $Y_j > 0$, positive consumption is called for even when the individual's net worth is negative, as long as it is greater than $-Y_j$ in Models I-III and greater than $-Y + x_j^o$ in Model IV. Only *at* these end points would the individual consume nothing.

Since $x_j + Y_j$ may be viewed as permanent (normal) income and since consumption is proportional $(0 < B_j \leq 1)$ to $x_j + Y_j$ in Models I-III, we see that the optimal consumption functions in these models satisfy the permanent (normal) income hypotheses precisely ([15], [9], and [7]).

In each model, $c_j^*(x_j)$ is decreasing in α. Thus, the greater the individual's impatience $(1-\alpha)$, the greater his present consumption would be. This, of course, is what we would expect.

By (19) and (20), the relative and absolute risk aversion indices of Models I-IV are as follows:

(a) $q^*(c) = 1 - \gamma$ Model I,

(b) $q^*(c) = 1 + \gamma$ Model II,

(c) $q^*(c) = 1$ Model III, and

(d) $q(c) = \gamma$ Model IV.

In Models I and II, it was found in [11] that a change in relative risk aversion may either decrease or increase present consumption. In Model IV, on the other hand, $c_j^*(x)$ is increasing in γ; i.e., a more risk-averse individual consumes more, *ceteris paribus*.

From (25) and (32) we observe that k_j is a natural measure of the "favorableness" of the investment opportunities in period j. This is because k_j is a maximum determined by (the one-period utility function and) the distribution function F_j; moreover, F_j is reflected in the solution only through k_j, and $f_j(x_j)$ is increasing in k_j. In [11] it is shown that the propensity to consume is *decreasing* in k_j in the case of Model I and *increasing* in k_j in Model II. In Model III, on the other hand, the optimal consumption strategy was found to be *independent* of the investment opportunities in every respect. While the marginal propensity to consume is independent of k_j in Model IV also, the *level* of consumption is in this case an increasing function of k_j. Thus, the class of utility functions we have examined implies an exceptionally rich pattern of consumption behavior with respect to the "favorableness" of investment opportunities.

In each case, we find that lending is linear in wealth. Turning first to Models I-III, we find that borrowing always takes place at the lower end of the wealth scale; (23) evaluated at $x_j = -Y_j$ gives $-Y_j < 0$ as the optimal amount to lend. From (23) we also find that $z_{1j}^*(x_j)$ is increasing in x_j if and only if $1-v_j^* > 0$ since $1-B_j^*$ is always positive. As a result, the models always call for borrowing at least when the individual is poor; whenever $1-v_j^* > 0$, they also always call for lending when he is sufficiently rich.

In Model IV, lending is always increasing in x_j. Thus, when an individual in this model becomes sufficiently wealthy, he will always become a lender.

In a very real sense, the properties exhibited by the optimal investment strategies form the cornerstone of the present paper. Turning first to Model IV, we note that the portfolio of productive investments is constant, both in mix and amount, at all levels of wealth. The optimal portfolio is also independent of the noncapital income stream and the level of impatience $1-\alpha$ possessed by the individual.

Similarly, we find in Models I-III that since for all i,m > 1 $z^*_{ij}(x)/z^*_{mj}(x) = v^*_{ij}/v^*_{mj}$ (which is independent of x_j, Y_j, J, and α), the *mix* of risky investments is independent of wealth, noncapital income, age, and impatience to spend. In addition, the *size* of the total investment commitment in each period is proportional to $x_j + Y_j$. We also note that when $Y_j = 0$, the ratio that the risky portfolio

$$\sum_{i=2}^{M_j} z^*_{ij}(x_j)$$

bears to the total portfolio

$$\sum_{i=1}^{M_j} z^*_{ij}(x_j)$$

is independent of wealth in each model.

In summary, then, we have the important result that the *optimal mix of risky (productive) investments* in each model is independent of the individual's wealth x_j, noncapital income stream $(y_j,\ldots,y_J)$, age, and rate of impatience to consume $1-\alpha$. The optimal mix depends in each period only on the probability distribution F_j of the returns for that period, the interest rate r_j-1 for that period, and the individual's one-period utility function of consumption u(c). As noted, this result is the starting point for the balance of the present study.

V. Bases for the Formation of Firms

We now state the following:

DEFINITION: The sequences

$$\langle F_j^{(1)} \rangle \equiv F_j^{(1)}, F_{j+1}^{(1)}, \ldots, F_{J_1}^{(1)}$$

and

$$\langle F_j^{(2)} \rangle \equiv F_j^{(2)}, F_{j+1}^{(2)}, \ldots, F_{J_2}^{(2)},$$

where $J_1, J_2 \geq j$, $J_1 \leq J_2$ are identical (express the same probability beliefs) if $F_k^{(1)} = F_k^{(2)}$, $k = j,\ldots,J_1$; otherwise, they are nonidentical. The infinite sequence $F_j, F_{j+1},\ldots$ will be written $\langle F_j \rangle$.

In view of (24), (30), and the preceding definition, the following consequence of Theorems 1 and 2 is immediate:

COROLLARY: Given the antecedents of Theorems 1 and 2 and a collection of individuals whose risk-aversion indices $q^*(c)$ or $q(c)$ (see (19) and (20)) are positive constants, there is a basis for the formation of firms, one for each nonidentical pair $(q^*, \langle F_j \rangle)$ and $(q, \langle F_j \rangle)$, such that individuals with the same risk-aversion index and the same probability beliefs may delegate the choice and mix of productive (risky) investments to the same firm for all future time regardless of each individual's wealth, noncapital income, age, and impatience to spend.

By the words "may delegate," we mean that the individual would be indifferent between making the risky investments himself and turning the total amount allocated to risky investments over to a firm for investment in return for an equity interest in the firm in the same amount. In line with our assumptions concerning the investment opportunities (see Section III), we assume that an individual can, at each decision point, increase or decrease his equity interest in a firm without cost. This may be accomplished by entering into a transaction with the firm (capital addition or withdrawal) or by effecting a cash exchange with one or more individuals. The firm $(q^*, \langle F_j \rangle)$ (or $(q, \langle F_j \rangle)$) will be said to be compatible with individuals whose risk-aversion indices are q^* (or q) and whose probability beliefs $\langle F_j^{(n)} \rangle$ are identical to those of $\langle F_j \rangle$. Similarly, the firm $(q^*, \langle F_j \rangle)$ (or $(q, \langle F_j \rangle)$) will be said to be *compatible in the* m^{th} *period* with individuals whose risk-aversion indices are q^* (or q) and whose probability beliefs with respect to the m^{th} period, F_m, coincide with member F_m in $\langle F_j \rangle$.

This result may be interpreted in two different ways. From a normative point of view, it indicates that there is a rational basis for economic cooperation (or, more accurately in our framework, no rational reason for noncooperation) among individuals with different goals and in different economic circumstances. Descriptively, the corollary suggests that the economic cooperation we observe among unlike individuals in the real world is consistent with, and may possibly have arisen as a result of, each individual's (selfish) desire to maximize expected utility from consumption over time.

VI. The Firm's Objective and Its Optimal Capital Structure

We shall now continue on the path just begun. The next implication which we shall state formally is:

THEOREM 3: Given the antecedents of Theorems 1 and 2 and a firm in the sense of the corollary, the objective of the firm may be stated as: In each period j, invest proportion v_{ij}^*/v_j^* of all capital (assets) $\bar{x}_j$ in activity i, $i = 2,\ldots,M_j$, where

$$v_j^* \equiv \sum_{i=2}^{M_j} v_{ij}^*$$

and the v_{ij}^* are the values of v_{ij} which maximize

(34) $$E\left\{u\left[\sum_{i=2}^{M_j}(\beta_{ij}-r_j)v_{ij} + r_j\right]\right\}$$

subject to

(26) $$v_{ij} \geq 0 \qquad i \notin S_j$$

and

(27) $$\Pr\{\sum_{i=2}^{M_j}(\beta_{ij}-r_j)v_{ij} + r_j \geq 0\} = 1$$

in the case of firm $(q^*, \langle F_j \rangle)$, or which maximize

(35) $$E\left\{u\left[\sum_{i=2}^{M_j}(\beta_{ij}-r_j)v_{ij}\right]\right\}$$

subject to (26) in the case of firm $(q,\langle F_j \rangle)$ (assuming, in each case, that $v_j^* > 0$). Moreover, when the firm has unlimited liability, the optimal capital structure (the ratio between debt and equity capital) of the firm is arbitrary.

PROOF: The first part of the theorem follows immediately from (24), (25), (26), (27), (30), and (32). Turning to the second part, let the individual's optimal investment strategy call for allocating, in a given period, amount

$$a_1 \equiv \sum_{i=2}^{M_j} z_{ij}^*(x_j)$$

to risky investments and

$$a_2 \equiv z_{1j}^*(x_j)$$

to lending (borrowing when a_2 is negative). Let the final debt-equity ratio[6] of the firm, i.e., the ratio after the owners have adjusted their capital positions at the decision point in question, be $\theta > -1$. By investing $a_1/(1+\theta)$ of equity capital in the firm, the individual then obtains the benefit of

[6]When $\theta < 0$, $-\theta$ may be referred to as the savings-equity ratio (see Section VIII).

$(1+\theta)a_1/(1+\theta) = a_1$ invested in the risky investments. If the individual lends the difference $a_1 - a_1/(1+\theta)$, his lending becomes $a_2 + a_1 - a_1/(1+\theta)$. But, his "share" of the debt of the firm is $\theta a_1/(1+\theta)$; his "net" lending is therefore $a_2 + a_1 - a_1/(1+\theta) - \theta a_1/(1+\theta) = a_2$. Hence, θ may be chosen arbitrarily.

Thus, starting with a collection of heterogeneous individuals, each of whom is intent on maximizing (his own) utility from consumption over time, we have not only found that there exists a basis for the formation of firms by *subcollections* of individuals (each subcollection in turn possessing significant heterogeneity), but also that each such firm has a well-defined (unique) objective function and that its capital structure (debt-equity ratio) is unimportant. The firm's objective function may be said to call for "profit maximization" where the precise meaning of this term *under risk and with respect to time* has been induced from the most basic of inputs: the preferences, economic circumstances, and perception of opportunities of its owners. Note that the maximization of (34) or (35) may be viewed as the *short-run* objective of the firm but that, performed repeatedly, the short-run maximization process also yields the *long-run* objective.

Concerning the optimal capital structure of the firm, Theorem 3 has an interesting relation to Proposition I of Modigliani and Miller: "The market value of any firm is independent of its capital structure and is given by capitalizing its expected return at the rate σ_k appropriate to its class" [16, p. 268]. To the extent that this proposition may be interpreted to mean that the *optimal* capital structure of the firm is arbitrary, Theorem 3 does, of course, support the proposition.

Perhaps the chief practical significance of Theorem 3 lies in the fact that it is often easier for the firm to borrow money than for the individual to do so, especially when the latter's net worth is negative. As indicated in the proof, individuals have the opportunity to shift, without loss of utility, a large portion of their borrowing requirements to the firm. Whenever $a_2 < 0$ and θ is such that $[a_1 - a_1/(1+\theta)] > |a_2|$, we find that the individual should, instead of borrowing on his own account, become a lender. Clearly, his lending might take the form of a position in the bonds of one or several of the firms considered in this section.

When $Y_j = 0$ for all j for all owners, the ratio between the financial portfolio and the productive portfolio is constant for all x_j since

$$z^*_{1j}(x_j)/\sum_{i=2}^{M_j} z^*_{ij}(x_j) = (1-v^*_j)/v^*_j,$$

which is constant. Thus, the owners may in this case delegate *all* of their borrowing (or lending) requirements to the firm by choosing $\theta = v_j^* - 1$.

VII. Stock Issues and "Dividends"

Let $K_j > 0$ be the number of owners of firm $(q^*,\langle F_j\rangle)$ or firm $(q,\langle F_j\rangle)$ at the j^{th} decision point, and let d_j^k be the optimal investment of the k^{th} owner in the firm at the beginning of period j. Denoting, as before, the assets of the firm at the beginning of period j by $\bar{x}_j$ and the chosen debt-equity ratio for period j by θ_j, we must then have

$$\bar{x}_j = (1+\theta_j)\sum_{k=1}^{K_j} d_j^k . \tag{36}$$

If we now denote the equity position of each owner at the *end* of period j *per unit* of investment at the beginning of period j by G_j, we obtain, by Theorem 3

$$G_j = (1+\theta_j)\sum_{i=2}^{M_j} \beta_{ij} v_{ij}^* / v_j^* - r_j\theta_j \tag{37}$$

since

$$\frac{\bar{x}_j G_j}{1+\theta_j} = \frac{\bar{x}_j}{1+\theta_j}[(1+\theta_j)\sum_{i=2}^{M_j}\beta_{ij}v_{ij}^*/v_j^* - \theta_j r_j]. \tag{38}$$

The optimal decision rules (23), (24), (30), and (31), now become, for all j,

$$d_j^k(x_j^k) = \frac{(1-B_j^k)v_j^*(x_j^k + Y_j^k)}{1+\theta_j}, \tag{39}$$

$$z_{1j}^k(x_j^k) = (1-B_j^k)(1-v_j^*)(x_j^k + Y_j^k) - Y_j^k + \theta_j d_j^k(x_j^k), \tag{40}$$

$$d_j^k(x_j^k) = \frac{E_j v_j^*}{1+\theta_j} \text{ , and} \tag{41}$$

$$z_{1j}^k(x_j^k) = x_j^k - c_j^{*k}(x_j^k) - d_j^k(x_j^k) \tag{42}$$

where $z_{1j}^k(x_j^k)$ is the optimal lending strategy of individual k as an *investor* in the firm.

It is clear that if the value at decision point j of each unit of equity held in the firm at decision point $j-1$ is G_{j-1}, then the equity of owner k at the end of the $j-1^{th}$ period is $G_{j-1}d^k_{j-1}$. Since the optimal investment at the j^{th} decision point is $d^k_j(x^k_j)$, it follows that the action required by owner k at the j^{th} decision point with respect to his holdings in firm $(q^*,\langle F_j\rangle)$ or firm $(q,\langle F_j\rangle)$ is to increase his position by

(43) $$d^k_j(x^k_j) - G_{j-1}d^k_{j-1}.$$

If it were negative, (43) would indicate the (negative of the) amount of capital to be withdrawn, or owner k's "personalized" dividend. If

(44) $$\sum_{k=1}^{K_j} d^k_j(x^k_j) - G_{j-1}\sum_{k=1}^{K_{j-1}} d^k_{j-1}(x^k_{j-1})$$

is positive, (44) is clearly the net inflow of equity capital to the firm at decision point j. If (44) is negative, it may be viewed as the firm's dividend. Similarly,

(45) $$\theta_j\sum_{k=1}^{K_j} d^k_j(x^k_j) - r_{j-1}\theta_{j-1}\sum_{k=1}^{K_{j-1}} d^k_{j-1}(x^k_{j-1})$$

is readily seen to be the net increase in the firm's debt capitalization at decision point j.

VIII. The Debt of the Firm: Limited Liability

When the legal status of the firm is such that it has limited liability (e.g., a corporation), the amount the firm may borrow is clearly not arbitrarily large since lenders expect both interest and the repayment of principal with probability 1 (at least in our model). However, since they have first claim against the firm's assets, there is a limit up to which they should not hesitate to lend. Formally we obtain:

THEOREM 4: Given a firm like that in Theorem 3, but with limited liability, the optimal capital structure of the firm is arbitrary except that there is an upper limit on the firm's *feasible* debt-equity ratio imposed by the limit attached to its liability. The upper limit on the debt-equity ratio in period j is given by

(46)
$$\frac{b_j(\bar{v}_j^*)/v_j^*}{r_j - b_j(\bar{v}_j^*)/v_j^*},$$

where $b_j(\bar{v}_j^*)$ is the greatest lower bound on b_j such that

$$\Pr\{\sum_{i=2}^{M_j}\beta_{ij}v_{ij}^* < b_j\} > 0$$

and

$$\bar{v}_j^* \equiv (v_{2j}^*,\ldots,v_{M_j j}^*).$$

PROOF: Using (38), the greatest lower bound on the firm's net worth at the end of period j is

(47)
$$\sum_{k=1}^{K_j} d_j^k(x_j^k)\left\{\left[1 + \theta_j\right] b_j(\bar{v}_j^*)/v_j^* - \theta_j r_j\right\}.$$

Thus, it is necessary for (47) to be at least zero in order for the creditors to receive their due with probability 1. Setting (47) equal to zero and solving for θ_j we obtain (46).

It follows from the "no-easy-money condition" that $b_j(\bar{v}_j^*) < r_j$ since the firm can only exist if $v_{ij}^* \neq 0$ for at least one i and $b_j(\bar{v}_j^*) < r_j$ for all such $\bar{v}_j^*$. Note that when $v_j^* \geq 1$, the maximum debt-equity ratio is always non-negative. However, when $v_j^* < 1$ *and* $b_j(\bar{v}_j^*)/v_j^* > r_j$ or $b_j(\bar{v}_j^*) < 0$, we observe that (46) may be negative. In this case, it becomes necessary for the stockholders to shift some of their savings to the firm (see (23), (24), (30), and (31)) when the firm has limited liability.

It is readily seen that when $v_j^* > 1$ and $b_j(\bar{v}_j^*)$ is close to its upper limit r_j, the maximum debt-equity ratio will be relatively large, and vice versa. This may, in part, explain why public utilities, for which any losses are typically small, are often found to have a higher debt-equity ratio than, for example, small electronics companies, for which large losses are much more probable. This is because almost any selection process in which each nondegenerate interval on the feasible debt-equity ratio scale of the firm has a positive probability of being chosen would produce this result.[7]

[7]Since the optimal debt-equity ratio is arbitrary, any selection process that will choose a feasible ratio with the probability of 1 would, of course, also be optimal.

IX. The Growth of the Firm

We have seen that the life of firm $(q^*, \langle F_j \rangle)$ or firm $(q, \langle F_j \rangle)$ may be indefinite (infinite) and unrelated to the lives of its owners at any one time. This gives a basis for examining more closely the growth pattern of the firm in the "long" run.

Several authors have shown that when investment returns are stochastically independent over time and all proceeds from one period's decisions are reinvested the next period, maximization of the expected logarithm of the end-of-period capital in each period results in the firm's long-run capital position being greater than under any other policy with a probability that approaches 1 (as the number of periods increases)([14], [4], [5], and [6]). Put differently, a long-run capital maximization objective induces, under the preceding conditions, a short-run objective function of the following simple form: maximize $E[\log x]$, where x is the capital position at the end of the current period.[8] This objective is unchanged under a dividend policy which calls for a fixed (though not necessarily stationary) proportion of capital to be paid out at the end of each period [14, p. 146]. What lies behind this result is the fact that maximizing the expected logarithm of the end-of-period capital is equivalent to maximizing the expected growth rate in the given period [11].

Referring to (24) and (25), we see that an individual who obeys Model III always invests the capital available after the allotment to current consumption so as to maximize the expected growth rate of capital plus the present value of the noncapital income stream. Since $q^*(c) < 1$ in Model I, $q^*(c) > 1$ in Model II, and $q^*(c) = 1$ in Model III (see Section IV), the optimal investment policy of Model I may be said to be too daring, while that of Model II is too conservative to achieve maximum expected growth of capital. The investment policy of Model IV is more difficult to characterize. Corroborating these statements is the fact that the solvency constraints (14) are never binding in Models II and III (since (27) is never binding in those models). This means that the individual obeying either of Models II and III would never risk losing everything (and hence be forced to consume nothing the rest of his life), whereas this possibility exists in Model I [12]. Thus, economic survival is guaranteed by the investment policy

[8] The logarithmic utility function has been advocated for other reasons as well; see Bernoulli [3], Savage [19], and Arrow [1].

that maximizes the expected growth rate as well as the more conservative optimal investment strategies of Model II.

Note that although in the logarithmic case firm $(q^*, \langle F_j \rangle)(q^* = 1)$ invests so as to maximize the expected growth rate of nonconsumed capital for each stockholder *in conjunction* with his *own* lending or borrowing, the firm does not necessarily maximize the expected growth rate of *its* equity capital. Letting

$$d_j \equiv \sum_{k=1}^{K_j} d_j^k ,$$

we observe from Theorem 3 (with $u(\cdot) = \log(\cdot)$) that the firm invests $(1+\theta_j)d_j v_{ij}^* / v_j^*$ in opportunity i, $i = 2,\ldots,M_j$ and borrows $\theta_j d_j$ in accomplishing its objective. However, if the firm wished to maximize the expected growth rate of d_j, it should invest $d_j v_{ij}^*$ in opportunity i, $i = 2,\ldots,M_j$ and lend $d_j(1 - v_j^*)$. Clearly, choosing $\theta_j = v_j^* - 1$ makes the two policies unidentical -- and $\theta_j = v_j^* - 1$ is always a feasible debt-equity ratio even under limited liability (see (27) and Theorem 4). Consequently, the firm that wishes not only to invest optimally for its (Model III) stockholders but also to maximize the expected growth rate of stockholders' equity should adopt a debt-equity ratio of $v_j^* - 1$ (if negative, a savings-equity ratio of $1 - v_j^*$).

X. The Balanced Mutual Fund

One type of real-world firm which corresponds quite closely to the firm implied by the model considered in this paper is the balanced, no-load, open-end mutual fund. This is because in no-load, open-end investment companies, stockholders can purchase or withdraw shares, in any fraction, at asset value, by entering into a transaction with the fund itself. In addition, the opportunities considered by the mutual fund tend to be liquid, highly divisible, to have stochastically constant returns to scale, and to have negligible conversion costs.

The chief purpose of the so-called balanced mutual fund is to invest in such a (balanced) way that its shareholders should not have to make any risky investments except the purchase of shares in the fund itself [13, p. 398]. It follows from the corollary that any balanced mutual fund that wishes to sell its shares on this basis, even though all individuals are in different economic circumstances (i.e., their wealths and noncapital income streams are different), possess varying degrees of impatience, fall in different age brackets, and behave so as to maximize expected utility from consumption over time, may have a sizeable market. This market consists of all individuals whose one-period utility functions of consumption are identical, having one of the forms $u(c) = c^{\gamma}$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma > 0$, $u(c) = \log c$, and $u(c) = -e^{-\gamma c}$, $\gamma > 0$, assuming that U is of the form (1) in the first place, *and* who have the same probability beliefs with respect to the returns from risky investments.

A sensible objective for the balanced mutual fund attempting to attract a large number of investors in a world of rational but heterogeneous individuals, then, is that given in Theorems 3 and 4. Even if the possibility of differences (among funds) in probability assessments of returns are ignored, there is within the confines of these theorems a theoretical basis for the existence of an infinite number of different balanced mutual funds, each characterizable by a unique risk-aversion index q or q^* (as given by (20) and (19)). As noted in Section IX, any fund which seeks "maximum growth" should be expected to have a risk-aversion index $q^* = 1$; more conservative funds would be consistent with indices $q^* > 1$.

XI. Nonprofit Organizations

The basic models (P1 and P2) underlying this study, developed with the individual in mind, also appear directly applicable to certain nonprofit organizations. Consider, for example, the case of endowed educational institutions such as private universities. All utility of such organizations appears to be derived via operating expenditures, which, if the rental value of physical assets owned and utilized are included in this category, correspond to consumption in the case of an individual. The noncapital income stream consists in this case of such items as tuition and other fees, grants, and bequests. Furthermore, both financial and productive opportunities exist, and are usually taken advantage of, with respect to that part of the capital (endowment) not allocated to the current operating budget. Thus, the model considered in this paper appears to apply also to the basic decision problem facing the endowed educational organization; in fact, this is one application in which an infinite horizon seems quite appropriate.

Similarly, the models would be applicable to endowed charitable organizations. Here, utility is derived via donations, which, therefore, correspond to consumption in the case of the individual. In fact, the additivity property of

(1) appears almost harmless in this case since we would not expect much of a ratchet effect, e.g., with respect to the giving of a private foundation. The noncapital income stream would be represented by new contributions to the endowment, if any. Finally, unspent funds must be invested, consequently, the decision problem faced by the endowed foundation is of the same type as that which has been modeled in the case of the individual.

XII. Generalizations

The crucial property of the solutions to P1 and P2 which has been exploited in this paper with respect to the theory of the firm is the constancy of the ratios $z^*_{ij}(x_j)/z^*_{kj}(x_j)$, $i = 2,\ldots,M_j$, where $z^*_{kj}(x_j) \neq 0$, i.e., the fact that the optimal mix of risky assets in P1 and P2 depend only on the risk-aversion index q or q^*, the distribution function F_j, and the interest rate $r_j - 1$ in each period. As shown in [12], this property is retained under certain conditions in more general models of the individual.

The models considered in [12] differ from those used in this paper in three respects. First, the individual's lifetime is postulated to be a random variable; second, a utility function intended to represent the individual's bequest motive is introduced; and third, the individual is offered the opportunity to purchase term insurance on his life. In view of these modifications, the individual's objective is postulated, more generally, to be the joint maximization of expected utility from consumption as long as he lives *and* from the bequest left upon his death.

When closed-form solutions to these more general models exist, as they do under certain conditions for the one-period utility functions (15), (16), (17), the optimal mix of risky assets is independent not only of wealth, noncapital income, age, and impatience to consume, but *also* of the strength of the individual's bequest motive, the individual's survival probabilities and, in some instances, his insurance coverage. Thus, the theory developed in the present paper applies directly to these more general models also.

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