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OPTIMAL INVESTMENT AND CONSUMPTION STRATEGIES UNDER RISK, AN UNCERTAIN LIFETIME, AND INSURANCE*

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1. INTRODUCTION AND SUMMARY

IN A PREVIOUS ARTICLE [8], a normative model of the individual's economic decision problem under risk was presented. In addition, certain implications of the model with respect to individual behavior were deduced for the class of utility functions, $\sum_{j=1}^{\infty} \alpha^{j-1}u(c_j)$, $0 < \alpha < 1$, where c_j is the amount of consumption in period j, such that either the risk aversion index $-u''(x)/u'(x)$, or the risk aversion index $-xu''(x)/u'(x)$, is a positive constant for all $x \geq 0$. In a second paper [6], it was further shown that this model, developed with the individual in mind, also gives rise to an induced theory of the firm under risk for the same class of utility functions.

In the foregoing model, it was assumed that the individual's horizon was infinite (or known with certainty). In this paper, we consider the same basic model with three modifications. First, we postulate that the individual's lifetime is a random variable with a known probability distribution. Second, we introduce a utility function intended to represent the individual's bequest motive. Third, we offer the individual the opportunity to purchase insurance on his life. It is found that when some or all of these modifications are made, all of the more important properties possessed by the optimal consumption and investment strategies under a certain horizon are preserved, albeit only under special conditions.

In Section 2, the various components of the decision process are constructed. In the earlier model, the individual's objective was assumed to be the maximization of expected utility from consumption over time. Here, we postulate, more generally, that his objective is to maximize expected utility from consumption as long as he lives *and* from the bequest left upon his death. As before, the individual's resources are assumed to consist of an initial capital position (which may be negative) and a non-capital income stream. The latter, which may possess any time-shape, is assumed to be known with certainty and to terminate upon his death. In addition to insurance available at a "fair" rate, the individual faces both financial opportunities (borrowing and lending) and an arbitrary number of productive investment opportunities. The interest rate is presumed to be known but may have any time-shape. The returns from the productive opportunities are assumed to be random variables, whose probability distributions may differ from period to period but are assumed to satisfy the "no-easy-money" condition. While no limit is placed on borrowing, the individual is required to be solvent at the time of his death with probability 1, that is, all debt must be fully secured at all times.

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The components developed in Section 2 are assembled into formal models in Sections 3, 4, 6, 7, 8, and 9. The fundamental approach taken is that the portfolio composition decision, the financing decision, the consumption decision, and, where applicable, the insurance decision, are all analyzed in *one* model. The vehicle of analysis is discrete-time dynamic programming.

Sections 4, 6, 7, and 8 consider the four possible combinations of no bequest motive/bequest motive and no insurance/insurance. Explicit solutions are derived, where possible, for that class of one-period utility functions whose proportional risk aversion indices are positive constants, and are found to have the same form as when the horizon is known. A review of the properties and implications of these solutions is given in Section 5; it is noted that due to the solvency constraint, the solution does not always exist in this form for all functions in the class.

In Section 9, the amount of insurance to be purchased in each period is included among the decision variables. When this is done, the solution is found to be of the indicated form only under highly specialized conditions; the optimal insurance strategy is found to be linear increasing in the future installments of the non-capital income stream.

In Section 10, it is shown that the models developed in this paper give rise to an induced theory of the firm under risk, which may be viewed as an extension of the theory developed for the case in which the horizon is certain [6]. Finally, it is shown in Section 11 that when the premium charged is "fair", an individual can in most instances increase his expected utility by selling insurance to others. Thus, any given individual may be able to make himself better off both by the purchase of insurance on his own life and the sale of insurance on the lives of others. Furthermore, both a supply of and a demand for insurance will exist in an economy of individuals whose utility functions belong to the class examined.

2. ASSUMPTIONS AND NOTATION

In this section, the postulates concerning the individual's preferences, resources, and opportunities will be specified. As the various building blocks are introduced, we also give the notation to be used in the following sections.

2.1. *Resources and opportunities.* We assume that the individual has the opportunity to make decisions at discrete points, called decision points, which are equally spaced in time. The space of time intervening between the two adjacent decision points j and $j+1$ will be denoted period j.

Let $\bar{p}_j > 0$ be the individual's probability of dying in the j-th period, $j = 1, \cdots, n$, where $\sum_{j=1}^{n} \bar{p}_j = 1$; thus n is the last period in which death may occur. We now observe that

$$(1) \qquad p_{mj} \equiv \bar{p}_j / \sum_{k=m}^{n} \bar{p}_k \qquad m, j = 1, \cdots, n \, (m \leq j)$$

expresses the probability that the individual will pass away in period j *given* that he is alive at *the beginning* of period m.

We denote the amount of the individual's monetary (capital) resources at the j-th decision point, given that he is alive at that point, by x_j. In the event the individual passes away in period $j-1$, the amount of his resources at the end of that period will be termed his estate and will be designated x'_j.

We assume that the individual may also be the recipient of a non-capital income stream during all or part of his life-time. If the individual is alive at decision point j, he will be paid the (finite) installment pertaining to period j, $y_j \geq 0$, at the *end* of that period; if he is not alive, he will receive nothing. In this paper, we make the fairly strong assumption that the individual's *potential* non-capital income stream is exogenously determined and is known in advance. It may be thought of as consisting of the income from labor, pensions, unemployment compensation, etc.

We postulate that the individual faces both financial and productive opportunities in each period. The first of these is the opportunity to borrow or lend arbitrary amounts of money in each period at the riskless (finite) rate $r_j - 1 > 0$ on the condition that any borrowings (including interest) must be fully secured. The amount saved at decision point j will be denoted z_{1j}; negative z_{1j} will then indicate borrowing.

For convenience, we shall define

$$(2) \qquad Y_j \equiv \frac{y_j}{r_j} + \frac{y_{j+1}}{r_j r_{j+1}} + \cdots + \frac{y_n}{r_j \cdots r_n}, \qquad j = 1, \cdots, n,$$

where Y_j may be interpreted as the present value of the individual's potential non-capital income stream at the j-th decision point.

The productive opportunities faced by the individual consist of the possibility of making risky investments. Let the total number of different risky (productive) opportunities available to the individual at decision point j be $M_j - 1$, of which the first $S_j - 1 \leq M_j - 1$ may be sold short. A short sale will be defined as the opposite of a long investment, that is, if the individual sells opportunity i short in the amount θ, he will receive θ immediately (to do with as he pleases) in return for the obligation to pay the transformed value of θ at the end of the period. The net proceeds realized at the end of period j from each unit of capital invested in opportunity i at the beginning of that period will be denoted β_{ij}. Thus, returns to scale are assumed to be stochastically constant, all investments are assumed to be realized in cash at the end of each period, and taxes and conversion costs, if any, are assumed to be proportional to the amount invested. The amount invested in opportunity i, $i = 2, \cdots, M_j$, at the j-th decision point will be denoted z_{ij}, and is, as indicated earlier, a decision variable along with z_{1j}.

It will be assumed that the joint distribution functions F_j given by

$$(3) \quad F_j(x_2, x_3, \cdots, x_{M_j}) \equiv \Pr\{\beta_{2j} \leq x_2, \beta_{3j} \leq x_3, \cdots, \beta_{M_j j} \leq x_{M_j}\}, \quad j = 1, \cdots, n$$

are known and independent[1]. In addition, we shall postulate that the $\{\beta_{ij}\}$

[1] In real world situations, the individual would, of course, be forced to derive his own subjective probability distributions. Numerous descriptions of how this may be accomplished, on the basis of postulates presupposing certain consistencies in behavior, are available in the literature; see, for example, the accounts of Savage [14] and Marschak [11].

satisfy the following conditions:

$$0 \leq \beta_{ij} < \infty\,, \qquad i = 2, \cdots, M_j;\ j = 1, \cdots, n\,, \tag{4}$$

$$\Pr\left\{\sum_{i=2}^{M_j} (\beta_{ij} - r_j)\theta_{ij} < 0\right\} > 0 \tag{5}$$

for all j and all finite θ_{ij} such that $\theta_{ij} \geq 0$ for all $i > S_j$ and $\theta_{ij} \neq 0$ for at least one i. (5) is known as the "no-easy-money" condition for the case when the lending rate equals the borrowing rate [8]. This condition states that no combination of productive investment opportunities exists in any period which provides, with probability 1, a return at least as high as the (borrowing) rate of interest; no combination of short sales is available in which the probability is zero that a loss will exceed the (lending) rate of interest; and no combination of productive investments made from the proceeds of any short sale can guarantee against loss.

In some variants of the basic model the individual has the opportunity to purchase term insurance on his own life and to sell (purchase) term insurance on the lives of others in each period. Let $t_j \geq 0$ denote the premium paid by the individual at the j-th decision point for life insurance *on his own life* during period j. If the individual dies during this period, which by (1) has probability p_{jj} of happening, we assume that his estate will receive t_j/p_{jj} at the end of period j; if he is alive at decision point $j+1$, he will receive nothing. Since in this contract the mathematical expectation of the "return" equals the cost, we shall say that the insurance is available at a "fair" rate. We assume that insurance is issued only when $p_{jj} < 1$, i.e., at decision points $1, \cdots, n-1$.

We shall allow the possibility of contracting in advance for purchases of insurance on the individual's own life. Such an arrangement will be called an insurance contract. The unexpired portion of such a contract at decision point j will be denoted $(\bar{t}_j, \bar{t}_{j+1}, \cdots, \bar{t}_{n-1})$, where $\bar{t}_k/p_{kk}$ is the amount of insurance the individual will keep in force in period k given that he is alive at the k-th decision point (when the premium $\bar{t}_k$ is paid).

For convenience we define

$$\bar{T}_j \equiv \bar{t}_j + \frac{\bar{t}_{j+1}}{r_j} + \cdots + \frac{\bar{t}_{n-1}}{r_j \cdots r_{n-2}}\,, \qquad j = 1, \cdots, n-1\,. \tag{6}$$

We also assume that

$$\bar{t}_j \leq x_j + B_j\,, \qquad j = 1, \cdots, n-1\,, \tag{7}$$

where B_j denotes the maximum an individual may borrow at the j-th decision point on the security of his non-capital income stream and his insurance contract. Since no insurance can be issued at the n-th decision point, it is clear that

$$B_n = y_n/r_n\,, \tag{8}$$

and that

$$B_j = \min\left\{\frac{y_j + \bar{t}_j/p_{jj}}{r_j}, \frac{y_j - \bar{t}_{j+1} + B_{j+1}}{r_j}\right\}, \qquad j = 1, \cdots, n-1\,, \tag{9}$$

where, by assumption,

(10) $$\bar{t}_n = 0 \,.$$

The purchase and sale of insurance on the lives of others will be viewed as a subset of the productive opportunities. Each person will be assumed to give rise to a separate investment opportunity, the return of which is independent of the returns of all other opportunities.

2.2. *The Utility Function.* The amount spent on consumption in period j will be designated c_j. As indicated, c_j is a decision variable; in order to give it economic meaning, we require it to be nonnegative.

We now postulate that the individual's preference ordering at the beginning of period m, conditioned on the event that death occurs in period $k \geq m$, is representable by a numerical utility function U_{mk}. This utility function is defined on the Cartesian product of all possible consumption programs $(c_m, \cdots, c_k)$ and the amount of his estate x'_{k+1} at the end of period k; thus, the utility function is independent of the opportunities faced by the individual. We assume in this paper that the conditional utility function U_{mk} has the form

(11) $$U_{mk}(c_m, \cdots, c_k, x'_{k+1}) = \frac{1}{\alpha_{m-1}} \sum_{j=m}^{k} \left(\prod_{i=m-1}^{j-1} \alpha_i \right) u(c_j) + \alpha_m \cdots \alpha_{k-1} \delta_k g(x'_{k+1})$$
$$m, k = 1, \cdots, n (m \leq k) \,.$$

Implicit in this form is the assumption that preferences are independent over time. We shall call $u(c)$ the one-period utility function of consumption and $g(x')$ the utility function of bequests. The constant $\alpha_j > 0$ $(\alpha_0 \equiv 1)$ is the patience factor linking the (one-period) utility functions of periods j and $j+1$ given that the individual will be alive at decision point $j+1$. When $\alpha_j < 1$ $(\alpha_j \geq 1)$ we shall say that impatience (patience) prevails in period j with respect to period $j+1$. Similarly, the constant δ_j expresses the relative weight attached to bequests by the individual at decision point j, given that death will occur in period j. Since α_j and δ_j are constants, we note that the rate of patience, while dependent on time, is independent of the overall level of utility (see [10]).

We also postulate that the individual obeys the von Neumann-Morgenstern postulates [15];[2] accordingly, the individual's objective is to maximize the expected utility attainable from consumption over his life-time and the estate remaining and bequeathed upon his death.[3] We also assume that the individual always prefers more consumption to less in any period, i.e., that $u(c)$ is monotone increasing, and that the bequest function $g(x')$ is non-decreasing. Finally, we assume that the individual is risk averse with respect to consumption, which implies that $u(c)$ is strictly concave, and that $u(c)$ and $g(x')$ are twice differentiable.

The notation developed in the previous section is summarized below before we proceed to construct our basic model:

[2] We assume, however, that the continuity postulate has been modified in such a way as to permit unbounded utility functions.

[3] In congruence with this premise, we assume that the functions (11) are cardinal.

$\bar{p}_j$ probability of death in period $j(j \le n)$,

p_{mj} probability of death in period $j(\ge m)$, given that the individual is alive at the beginning of period m,

x_j capital position at decision point j,

x'_j estate at the end of period $j-1$, given that death occurs in period $j-1$,

y_j non-capital income received at the end of period j if the individual is alive at the beginning of period j,

Y_j present value at decision point j of the potential non-capital income stream,

$r_j - 1$ interest rate in period j,

z_{1j} amount lent at decision point j,

M_j number of investment opportunities in period j,

S_j number of investment opportunities which may be sold short in period j,

β_{ij} net proceeds realized at the end of period j from each unit invested in opportunity $i, i = 2, \cdots, M_j$, at the beginning of period j,

F_j joint distribution function of $\beta_{2j}, \cdots, \beta_{M_j j}$,

z_{ij} amount invested in opportunity $i, i = 1, \cdots, M_j$, at the beginning of period j,

t_j insurance premium paid at the beginning of period j for insurance in period j,

$\bar{t}_j$ contractual insurance premium payable at the beginning of period j if individual is alive at that point,

$\bar{T}_j$ present value at decision point j of potential premiums $\bar{t}_j, \bar{t}_{j+1}, \cdots, \bar{t}_{n-1}$,

c_j amount of consumption in period j,

U_{mk} utility function at the beginning of period m of consumption and bequests given that the individual passes away in period $k \ge m$,

u one-period utility function of consumption,

g utility function of bequests,

α_j patience factor linking periods j and $j+1$ if the individual remains alive at the end of period j,

δ_j patience factor linking periods j and $j+1$ if the individual passes away in period j.

3. DERIVATION OF THE BASIC MODEL

We shall now identify the relation which determines the amount of capital (debt) on hand at each decision point in terms of the amount on hand at the previous decision point. This leads to the pair of difference equations:

(12) $$x_{j+1} = \sum_{i=2}^{M_j} \beta_{ij} z_{ij} + r_j z_{1j} + y_j \, , \qquad j = 1, \cdots, n-1 \, ,$$

and

(13) $$x'_{j+1} = \sum_{i=2}^{M_j} \beta_{ij} z_{ij} + r_j z_{1j} + y_j + t_j / p_{jj} \, , \qquad j = 1, \cdots, n \, ,$$

where

$$z_{1j} = x_j - c_j - t_j - \sum_{i=2}^{M_j} z_{ij}\,, \qquad j = 1, \cdots, n \tag{14}$$

by direct application of the definitions given in Section 2.1. The first terms of (12) and (13) represent the proceeds from productive investments, the second terms the payment of the debt or the proceeds from savings, the third terms the non-capital income received, and the fourth term in (13) the proceeds from life insurance.

Inserting (14) into (12) and (13) we obtain

$$x_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij} - r_j)z_{ij} + r_j(x_j - c_j - t_j) + y_j\,, \qquad j = 1, \cdots, n-1\,, \tag{15}$$

and

$$x'_{j+1} = \sum_{i=2}^{M_j} (\beta_{ij} - r_j)z_{ij} + r_j(x_j - c_j - t_j) + y_j + t_j/p_{jj}\,, \qquad j = 1, \cdots, n\,. \tag{16}$$

The restriction that only the first S_j opportunities may be sold short in period j implies that

$$z_{ij} \geq 0\,, \quad i = S_j + 1, \cdots, M_j\,, \qquad j = 1, \cdots, n \tag{17}$$

must hold while the assumption that all borrowing must be fully secured implies that x'_j must satisfy the condition

$$\Pr\{x'_j \geq 0\} = 1\,, \qquad j = 2, \cdots, n+1\,. \tag{18}$$

By (5) it follows that there is an upper limit on consumption in period j, $j = 1, \cdots, n$, given by

$$x_j + B_j - \bar{t}_j\,, \qquad j = 1, \cdots, n\,, \tag{19}$$

which, since $c_j \geq 0$, must be non-negative in order that a feasible solution exist in period j.

We shall now define $f_j(x_j)$ as the maximum expected utility attainable by the individual over his remaining life-time, as of the beginning of period j, on the condition that he is alive at that point and that his capital is x_j. Utilizing (1) and (11), we may write this definition formally:

$$\begin{aligned} f_j(x_j) &\equiv \max E[p_{jj}U_{jj}(c_j, x'_{j+1}) + p_{j,j+1}U_{j,j+1}(c_j, c_{j+1}, x'_{j+2}) \\ &\qquad + \cdots + p_{jn}U_{jn}(c_j, \cdots, c_n, x'_{n+1})] \qquad j = 1, \cdots, n \\ &= \max E[u(c_j) + p_{jj}\delta_j g(x'_{j+1}) + \sum_{k=j+1}^{n} p_{jk}\alpha_j u(c_{j+1}) \\ &\qquad + p_{j,j+1}\alpha_j\delta_{j+1}g(x'_{j+2}) + \sum_{k=j+2}^{n} p_{jk}\alpha_j\alpha_{j+1}u(c_{j+2}) \\ &\qquad + \cdots + p_{jn}\alpha_j \cdots \alpha_{n-1}\delta_n g(x'_{n+1})]\,, \qquad j = 1, \cdots, n\,. \end{aligned} \tag{20}$$

By the principle of optimality, (20) may be written, using (1):[4]

[4] The principle of optimality states that an optimal strategy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision. See [2, (83)].

(22) $$f_j(x_j) = \max \{u(c_j) + E[\delta_j p_{jj} g(x'_{j+1}) + \alpha_j(1 - p_{jj}) f_{j+1}(x_{j+1})]\} , \quad j = 1, \cdots, n .$$

Letting

(23) $$a_j \equiv \delta_j p_{jj}$$

and

(24) $$b_j \equiv \alpha_j(1 - p_{jj}) ,$$

(22) may be written more concisely as

(25) $$f_j(x_j) = \max \{u(c_j) + E[a_j g(x'_{j+1}) + b_j f_{j+1}(x_{j+1})]\} , \qquad j = 1, \cdots, n .$$

We shall now attempt to obtain the solutions to (25) for certain classes of the functions $u(c)$ under different sets of assumptions concerning the bequest function $g(x')$ and the availability of insurance. More specifically, we shall consider the class of functions $u(c)$ such that $u(c)$ satisfies one (or more) of the functional equations

(26) $$u(xy) = v(x)w(y) ,$$

(27) $$u(xy) = v(x) + w(y) ,$$

for $c \geq 0$. The functional equations (26) and (27) in three unknowns belong to the set of equations usually referred to as the generalized Cauchy equations or Pexider's equations. That subset of their solutions, which is monotone increasing and strictly concave in u, is given (leaving out v and w) by [**9**]:

(28) $$u(c) = c^\gamma \qquad 0 < \gamma < 1 \qquad \text{Model I}$$

(29) $$u(c) = -c^{-\gamma} \qquad \gamma > 0 \qquad \text{Model II}$$

(30) $$u(c) = \log c \qquad \text{Model III} .$$

Note that since $u(c)$ is a cardinal utility function, the solutions (28)–(30) also include every solution $\lambda_1 + \lambda_2 u(c)$ to (26) and (27) where λ_1 and $\lambda_2 > 0$ are constants, if simultaneously, $g(x')$ is represented by $\lambda_2 g(x')$.

In [**9**], it was also noted that (28)–(30) is the solution to the differential equation

(31) $$cu''(c) + \gamma u'(c) = 0 \qquad \gamma > 0 .$$

Thus, (28)–(30) are also the only monotone increasing and strictly concave utility functions for which the proportional risk aversion index

(32) $$q^*(c) \equiv -cu''(c)/u'(c)$$

is a positive constant.

4. NO BEQUEST MOTIVE, NO INSURANCE

We shall first consider the simplest case, namely that in which there is no bequest motive and no insurance is available. The absence of a bequest motive implies that

(33) $$\delta_j g(x'_{j+1}) = 0 , \qquad j = 1, \cdots, n .$$

For (33) to hold, it is clearly necessary and sufficient either that $\delta_j = 0$ for all j or that $g(x')$ is identically zero. A utility function (11) for which (33) holds has been called a Fisher utility function [17]. The unavailability of insurance of course implies that

$$t_j = \bar{t}_j = 0\,, \qquad j = 1, \cdots, n-1\,, \tag{34}$$

which in turn, by (9) and (10), implies that

$$B_j = y_j/r_j\,, \qquad j = 1, \cdots, n\,. \tag{35}$$

Utilizing (23), (33), (15), (34), (18), and (16), and letting $\bar{z}_j \equiv (z_{2j}, \cdots, z_{M_j j})$, (25) now becomes

PROBLEM A.

$$f_j(x_j) = \max_{c_j, \bar{z}_j} \left\{ u(c_j) + b_j E\left[f_{j+1}\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j) + y_j \right)\right]\right\} \qquad j = 1, \cdots, n\,, \tag{36}$$

where

$$f_{n+1}(x_{n+1}) \equiv 0 \tag{37}$$

$$c_j \geq 0\,, \qquad j = 1, \cdots, n\,, \tag{38}$$

$$z_{ij} \geq 0\,, \quad i = S_j + 1, \cdots, M_j\,, \qquad j = 1, \cdots, n\,, \tag{17}$$

and

$$\Pr\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j) + y_j \geq 0 \right\} = 1 \qquad j = 1, \cdots, n. \tag{39}$$

Before attempting to obtain the solution to this problem, we shall state two preliminary results.

LEMMA: *Let $u(c)$, β_{ij}, and r_j be defined as in Section 2. Then the functions*

$$h_j(v_{2j}, \cdots, v_{M_j j}) \equiv E\left[u\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) v_{ij} + r_j \right)\right] \tag{40}$$

subject to

$$\Pr\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) v_{ij} + r_j \geq 0 \right\} = 1 \tag{41}$$

and

$$v_{ij} \geq 0\,, \qquad i = S_j + 1, \cdots, M_j \tag{42}$$

have (finite) maxima and the maximizing $v_{ij} (\equiv v_{ij}^*)$ are bounded and unique for all i and j.

COROLLARY: *Let $u(c)$, β_{ij}, and r_j be defined as in Section 2. Moreover, let $u(c)$ be such that it has no lower bound. Then the vector $\bar{v}_j^* \equiv (v_{2j}^*, \cdots, v_{M_j j}^*)$ which maximizes (40) subject to (41) and (42) is interior with respect to (41), that is*

$$\Pr\left\{\sum_{i=2}^{M_j} (\beta_{ij} - r_j)v_{ij}^* + r_j > 0\right\} = 1 .$$

The proofs of the Lemma and of the Corollary were given in [8] and will therefore be omitted. In the same paper, it was also shown that if $b(\bar{v}_j)$ denotes the greatest lower bound on b such that

$$\Pr\left\{\sum_{i=2}^{M_j} (\beta_{ij} - r_j)v_{ij} < b\right\} > 0$$

then $b(\bar{v}_j) < 0$ for $\bar{v}_j \neq 0$ and $b(0) = 0$.

We shall now prove the following result.

THEOREM 1: *Let $b_j, \beta_{ij}, r_j, F_j, y_j$ and Y_j be defined as in Sections* 2 *and* 3. *Moreover, let $u(c_j)$ be one of the functions* (28), (29) *or* (30), *let v_{ij}^* ($v_j^* \equiv \sum_{i=2}^{M_j} v_{ij}^*$) be the values of v_{ij} which maximize* (40) *subject to* (41) *and* (42) *and let F_k be such that $b(\bar{v}_k^*) > -r_k$ for $k = j, \cdots, m$ where m is the largest k for which $Y_{k+1} > 0$. Then, for x_j such that*

(43a) $$x_j \geq -Y_j , \qquad j > m$$

(43b) $$x_j \geq \max\{\underline{\underline{x}}_j, \underline{\underline{x}}_j'\} \qquad j \leq m$$

where

(44) $$\underline{\underline{x}}_k = \frac{\bar{\underline{\underline{x}}}_{k+1} + Y_{k+1}}{(1 - N_k)(b(\bar{v}_k^*) + r_k)} - Y_k , \qquad k = j, \cdots, m ,$$

(45) $$\underline{\underline{x}}_k' = \frac{Y_{k+1}}{(1 - N_k)(b(\bar{v}_k^*) + r_k)} - Y_k , \qquad k = j, \cdots, m ,$$

$$\bar{\underline{\underline{x}}}_k = \max\{\underline{\underline{x}}_k, \underline{\underline{x}}_k'\} , \qquad k = j + 1, \cdots, m ,$$

and

$$\bar{\underline{\underline{x}}}_{m+1} = -Y_{m+1} ,$$

the solution to Problem A *is given by, for $j = 1, \cdots, n$,*

(46) $$f_j(x_j) = K_j u(x_j + Y_j) + L_j$$

(47) $$c_j^*(x_j) = N_j(x_j + Y_j)$$

(48) $$z_{ij}^*(x_j) = (1 - N_j)v_{ij}^*(x_j + Y_j) , \qquad i = 2, \cdots, M_j ,$$

(49) $$z_{1j}^*(x_j) = (1 - N_j)(1 - v_j^*)x_j - Y_j(N_j(1 - v_j^*) + v_j^*) .$$

Letting

(50) $$k_j \equiv E\left[u\left(\sum_{i=2}^{M_j}(\beta_{ij} - r_j)v_{ij}^* + r_j\right)\right], \qquad j = 1, \cdots, n$$

the constants K_j, L_j, and N_j are given by

(a) *in the case of* Model I:

(51) $$\begin{aligned} K_j &= [1 + (b_j k_j)^{1/(1-\gamma)} + (b_j b_{j+1} k_j k_{j+1})^{1/(1-\gamma)} + \cdots \\ &\quad + (b_j \cdots b_{n-1} k_j \cdots k_{n-1})^{1/(1-\gamma)}]^{1-\gamma} \\ &= A_j^{1-\gamma} \end{aligned}$$

$$L_j = 0 \tag{52}$$

$$N_j = A_j^{-1}. \tag{53}$$

(b) *in the case of* Model II:

$$\begin{aligned} K_j &= [1 + (b_j(-k_j))^{1/(\gamma+1)} + (b_j b_{j+1}(-k_j)(-k_{j+1}))^{1/(\gamma+1)} + \cdots \\ &\quad + (b_j \cdots b_{n-1}(-k_j) \cdots (-k_{n-1}))^{1/(\gamma+1)}]^{\gamma+1} \\ &= A_j^{1+\gamma} \end{aligned} \tag{54}$$

$$L_j = 0 \tag{55}$$

$$N_j = A_j^{-1}. \tag{56}$$

(c) *in the case of* Model III:

$$K_j = 1 + b_j + b_j b_{j+1} + \cdots + b_j \cdots b_{n-1} \tag{57}$$

$$\begin{aligned} L_j = &-K_j \log K_j + (b_j + \cdots + b_j \cdots b_{n-1})(k_j + \log b_j) \\ &+ (b_j b_{j+1} + \cdots + b_j \cdots b_{n-1})(k_{j+1} + \log b_{j+1}) + \cdots \\ &+ b_j \cdots b_{n-1}(k_{n-1} + \log b_{n-1}) \end{aligned} \tag{58}$$

$$N_j = K_j^{-1}. \tag{59}$$

Furthermore, the solution is unique.

PROOF: For $j = n$, the proof is trivial. Let us therefore turn to the case when $j < n$.

Since k_j is finite for all j by the Lemma, it immediately follows that the constants K_j, L_j, and N_j are finite. Let $T_j(x_j)$, $j = 1, \cdots, n-1$, denote the right hand side of (36) upon inserting (46) (with subscript $j+1$) for $f_{j+1}(x_{j+1})$ in the case of Model I. This gives

$$T_j(x_j) = \max_{c_j, \bar{z}_j} \left\{ c_j^\gamma + b_j E\left[A_{j+1}^{1-\gamma}\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j) + y_j + Y_{j+1} \right)^\gamma \right] \right\},$$

which may be written

$$\begin{aligned} T_j(x_j) = \max_{c_j, \bar{z}_j} \Bigg\{ & c_j^\gamma + b_j A_{j+1}^{1-\gamma}(x_j - c_j + Y_j)^\gamma \\ & E\left[\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) \frac{z_{ij}}{x_j - c_j + Y_j} + r_j \right)^\gamma \right] \Bigg\} \end{aligned} \tag{60}$$

since (39), by the fact that $b(\bar{z}_j) \leq 0$ and $y_j / r_j \leq Y_j$, implies that

$$x_j - c_j + Y_j \geq 0 \tag{61}$$

and, because $b(\bar{z}_j)$ is decreasing in $\bar{z}_j$ [8],

$$\Pr\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) \frac{z_{ij}}{x_j - c_j + Y_j} + r_j \geq 0 \right\} = 1 \tag{62}$$

for all feasible c_j and $\bar{z}_j$.

Let us now assume that the constraint (39) is not binding, that is, that the solution is unaffected by its presence. By (40), the last factor in (60) may be written

$$h_j\left[\frac{z_{2j}}{x_j - c_j + Y_j}, \cdots, \frac{z_{M_j j}}{x_j - c_j + Y_j}\right],$$

where

(63)
$$\frac{\partial T_j}{\partial h_j} \geq 0 ,$$

with equality holding only when it holds in (61), in which case only $\bar{z}_j = (0, \cdots, 0)$ is feasible since $b(\bar{z}_j) < 0$ for $\bar{z}_j \neq 0$. This follows from (61) and the fact that $b_j > 0$ and $A_{j+1} > 0$; the latter assertion follows from (51) since $k_j \geq r_j^{\gamma} > 0$ by the fact that the zero solution is always feasible in (40). Since the maximum of h_j subject to (41) and (42) is k_j and the strategy

(64)
$$z_{ij}(x_j) = v_{ij}^*(x_j + Y_j - c_j) , \qquad i = 2, \cdots, M_j$$

assures this value for any feasible c_j, (64) is clearly the maximizing strategy regardless of the values of x_j and c_j when (39) is not binding. (60) now becomes

(65)
$$T_j(x_j) = \max_{c_j} \{c_j^{\gamma} + k_j b_j A_{j+1}^{1-\gamma}(x_j - c_j + Y_j)^{\gamma}\} .$$

Solving

$$\frac{\partial T_j}{\partial c_j} = \gamma c_j^{\gamma-1} - \gamma k_j b_j A_{j+1}^{1-\gamma}(x_j - c_j + Y_j)^{\gamma-1} = 0$$

for c_j we obtain (47). Since $T_j(x_j)$ is the sum of two strictly concave functions, it is itself strictly concave for $0 \leq c_j \leq x_j + Y_j$. Consequently, since $0 < N_j < 1$, (see (70)), $c_j^*(x_j)$, as given by (47), is unique. Inserting (47) in (64), (64) becomes (48) so that $(z_{2j}^*(x_j), \cdots, z_{M_j j}^*(x_j))$ is then also unique.

Thus,

$$\begin{aligned} T_j(x_j) &= N_j^{\gamma}(x_j + Y_j)^{\gamma} + k_j b_j A_{j+1}^{1-\gamma}(x_j + Y_j)^{\gamma}(1 - N_j)^{\gamma} \\ &= \left(A_j^{-\gamma} + (A_j - 1)^{1-\gamma}\left(1 - \frac{1}{A_j}\right)^{\gamma}\right)(x_j + Y_j)^{\gamma} \\ &= A_j^{1-\gamma}(x_j + Y_j)^{\gamma} \\ &= f_j(x_j) . \end{aligned}$$

We must now show that the preceding solution exists, without violating (39), for finite x_j. Letting $\underline{x}_{j+1}(\underline{x}'_{j+1})$ denote the greatest lower bound on the capital position (estate) at decision point $j + 1$, we obtain, upon insertion of (47) and (48) in (16) and (15)

(66)
$$\underline{x}'_{j+1} = (x_j + Y_j)(1 - N_j)(b(\bar{v}_j^*) + r_j) - Y_{j+1} , \qquad j = 1, \cdots, n ,$$

and

(67)
$$\underline{x}_{j+1} = \underline{x}'_{j+1} , \qquad j = 1, \cdots, n - 1 .$$

But since $f_j(x_j)$ is given by (46) only if $f_{j+1}(x_{j+1})$ is (with subscript $j + 1$) and

$\underline{\underline{x}}'_{j+1} \geq 0$, $f_{j+1}(x_{j+1})$ is of the form (46) only if $\underline{\underline{x}}'_{j+2} \geq 0$, etc., so that x_j must be sufficiently large for $\underline{\underline{x}}'_{j+1}, \underline{\underline{x}}'_{j+2}, \cdots, \underline{\underline{x}}'_{n+1}$ to be nonnegative under strategies (47), (48) and (49).

By (50), we note that $k_j \geq u(r_j)$ since $h_j(0, \cdots, 0) = u(r_j)$. Thus, by the Lemma,

(68) $$r_j^\gamma \leq k_j < \infty \qquad \text{Model I}$$

(69) $$-r_j^{-\gamma} \leq k_j < 0 \qquad \text{Model II}$$

$$\log r_j \leq k_j < \infty \qquad \text{Model III}.$$

As a result, we observe from (53), (56), and (59) that

(70) $$0 < N_1 < N_2 < \cdots < N_n = 1 .$$

Consider all decision-points k such that $k \geq m + 2$, i.e., such that $Y_k = 0$. Then, since

$$(1 - N_j)(b(\bar{v}_j^*) + r_j) \geq 0 , \qquad j = 1, \cdots, n$$

by (70) and (42), it follows from (66) and (67) that it is necessary and sufficient for $\underline{\underline{x}}'_k \geq 0$, $k = m + 2, \cdots, n + 1$, that

(71) $$x_{m+1} \geq -Y_{m+1} ,$$

which implies that (39) is not binding if

(43a) $$x_j \geq -Y_j , \qquad j > m .$$

By (71) and (39) we now find, when $j = m$, that it is necessary and sufficient for the solution to be of the given form that

$$\underline{\underline{x}}_{m+1} \geq \bar{\underline{\underline{x}}}_{m+1} = -Y_{m+1}$$

and

$$\underline{\underline{x}}'_{m+1} \geq 0 ,$$

so that we must have, by (66) and (67),

$$x_m \geq \bar{\underline{\underline{x}}}_m = \max\{\underline{\underline{x}}_m, \underline{x}'_m\}$$

where

$$\underline{\underline{x}}_m = \frac{\bar{\underline{\underline{x}}}_{m+1} + Y_{m+1}}{(1 - N_m)(b(\bar{v}_m^*) + r_m)} - Y_m ,$$

$$\underline{\underline{x}}'_m = \frac{Y_{m+1}}{(1 - N_m)(b(\bar{v}_m^*) + r_m)} - Y_m .$$

Repeating this process, we obtain that the solution is such that (39) is not binding if

(43b) $$x_j \geq \max\{\underline{\underline{x}}_j, \underline{\underline{x}}'_j\} , \qquad j \leq m .$$

By (70) and the assumption that

(72) $$b(\bar{v}_k^*) + r_k > 0 , \qquad k = j, \cdots, m ,$$

it then follows, from (44) and (45), that the right-hand side of (43b) is finite. Thus a solution for which (39) is not binding exists whenever (72) holds, i.e., the solutions to (40) are interior with respect to (41), $k = j, \cdots, m$. This concludes the proof for Model I; the proof for Models II and III is analogous. Note, however, that by the Corollary

$$b(\bar{v}_j^*) + r_j > 0 , \qquad j = 1, \cdots, n$$

always holds in the case of Models II and III.

5. PROPERTIES OF THE OPTIMAL STRATEGIES

It is easily demonstrated that the case in which the individual's life-time is certain is readily retrieved from Theorem 1. Assuming that the individual's death occurs in period n, the decision problem is identical to Problem A except that b_j must be replaced by α_j, $j = 1, \cdots, n-1$, and (39) must be replaced by

$$\text{(73)} \qquad \Pr\left\{\sum_{i=2}^{M_j}(\beta_{ij} - r_j)z_{ij} + r_j(x_j - c_j) + y_j \geq -Y_{j+1}\right\} = 1 , \qquad j = 1, \cdots, n ,$$

since the individual will be solvent at decision point k if $x_k \geq -Y_k$. Since (73) is always satisfied under (the optimal) strategies (47), (48), and (49) whenever $x_j \geq -Y_j$, the solution is given by Theorem 1 with b_j replaced by α_j and exists for $x_j \geq -Y_j$ (see also [8]). The introduction of an uncertain life-time, then, has two effects. First, since $b_j < \alpha_j$ for $j < n$ by (24), it is equivalent to a *decrease* in the patience rate α_j. Second, it reduces the individual's borrowing power whenever he has a non-capital income stream extending beyond the current period, the effect of which is to change the structure of the optimal solution (in an unknown fashion) for the lower levels of wealth.

From (5) and (16) we obtain that it is sufficient, but not necessary, that $z_{ij}^*(x_j) \neq 0$ for some $i \geq 2$ in order that

$$\Pr\{x'_{j+1} > 0\} > 0 , \qquad j = 1, \cdots, n-1 .$$

Furthermore, (47) and (48) imply that

$$\Pr\{x'_{n+1} = 0\} = 1$$

as noted earlier. Thus, should the individual pass away prior to the n-th period, there is a good chance that he will leave an estate even though he has no bequest motive. There are two factors which contribute to this; first, his concern for future consumption should he be alive at the end of the current period; and second, the possibility that the return from his current investments will exceed the infimum return.

As indicated, the solution to Problem A has essentially the same structure as the solution for the case in which the horizon is known [8]. Thus, we note, for example, that the optimal consumption strategy (47) satisfies the properties of the permanent (normal) income hypothesis [5], [12], [3]. From (70), we also observe that the constant of proportionality, N_j, increases with age.

Since

$$\frac{\partial N_j}{\partial b_k} < 0\,, \qquad j = 1, \cdots, n-1\,, \qquad k = j, \cdots, n-1$$

we obtain, using (24), that

$$\frac{\partial N_j}{\partial \alpha_k} < 0\,, \qquad j = 1, \cdots, n-1\,, \qquad k = j, \cdots, n-1$$

which implies that the greater an individual's impatience is with respect to some future period, the greater his present consumption would be.

By (32), we obtain

$$q^*(c) = 1 - \gamma \tag{74}$$

$$q^*(c) = \gamma + 1 \tag{75}$$

$$q^*(c) = 1 \tag{76}$$

for Models I, II and II, respectively. In the case of Model I, we observe from (53) that a sufficient condition for

$$\frac{\partial N_j}{\partial(1-\gamma)} > 0\,, \qquad j = 1, \cdots, n-1$$

to hold is that $b_j k_j \geq 1$ and $b(\bar{v}_j^*) \geq 1 - r_j$ for $j = 1, \cdots, n-1$. When these conditions are violated, the sign of $\partial N_j/\partial(1-\gamma)$ is ambiguous. The situation in Model II is similar: it is sufficient for

$$\frac{\partial N_j}{\partial(1+\gamma)} > 0\,, \qquad j = 1, \cdots, n-1$$

to hold that $b_j(-k_j) \geq 1$ and $b(\bar{v}_j^*) \geq 1 - r_j$ for $j = 1, \cdots, n-1$; otherwise the sign of $\partial N_j/\partial(1+\gamma)$ is ambiguous. Thus, it is not necessarily true that the more risk aversive the individual, is the more he will favor present consumption at the expense of future consumption.

A natural measure of the "favorableness" of the investment opportunities in the j th period is given by k_j (see (50)). Since in the case of Model I, by (53),

$$\frac{\partial N_j}{\partial k_m} < 0\,, \qquad j = 1, \cdots, n-1; m = j, \cdots, n-1\,,$$

while for Model II (56) gives

$$\frac{\partial N_j}{\partial k_m} > 0\,, \qquad j = 1, \cdots, n-1; m = j, \cdots, n-1\,,$$

we find that the propensity to consume is decreasing in k_m in Model I and increasing in k_m in Model II. In Model III, we observe from (59) that the optimal consumption strategy is independent of the investment opportunities in every respect.

Turning to (49), we find that the optimal lending strategy is linear in wealth. For $j < n, z_{1j}^*(x_j)$ is clearly increasing in wealth if and only if $1 - v_j^* > 0$ and decreasing if and only if $1 - v_j^* < 0$ since, for such $j, 1 - N_j > 0$ by (70).

Proceeding to (48), we find that for any m and j,

$$\frac{z_{ij}^*(x_j)}{z_{mj}^*(x_j)} = \frac{v_{ij}^*}{v_{mj}^*}, \qquad i = 2, \cdots, M_j .$$

Since the v_{ij}^*, $i = 2, \cdots, M_j$, are constants which depend only on F_j, r_j, and $u(.)$, i.e., the probability distribution of the returns, the interest rate, and the individual's one-period utility function of consumption, it is apparent that in each period the optimal *mix* of risky (productive) investments in each of Models I—III is independent of the individual's wealth, non-capital income stream, age, and rate of impatience to consume. Moreover, the *size* of the total investment commitment in risky opportunities is in each period proportional to $x_j + Y_j$, that is, the individual's wealth plus the present value of his *potential* non-capital income stream. We also note that when $Y_j = 0$, the ratio that the risky portfolio $\sum_{i=2}^{M_j} z_{ij}^*(x_j)$ bears to the total portfolio $\sum_{i=1}^{M_j} z_{ij}^*(x_j)$ is independent of wealth, age, and impatience in each model.

6. BEQUEST MOTIVE, INSURANCE UNAVAILABLE

We shall now consider the case when there is a bequest motive but no insurance is available. Were it not for the presence of the functions $a_j g(x')$, this problem would clearly be identical to Problem A. Utilizing (25), (15), (16), (34), and (18) we obtain

PROBLEM B.

$$\begin{aligned} f_j(x_j) = \max_{c_j, z_j} \Big\{ u(c_j) + E\Big[a_j g\Big(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} \\ + r_j(x_j - c_j) + y_j \Big) \Big| + b_j f_{j+1}\Big(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} \\ + r_j(x_j - c_j) + y_j \Big) \Big] \Big\} \end{aligned} \qquad j = 1, \cdots, n , \tag{77}$$

where

$$f_{n+1}(x_{n+1}) \equiv 0 \tag{37}$$

$$c_j \geq 0 , \qquad j = 1, \cdots, n , \tag{38}$$

$$z_{ij} \geq 0 , \quad i = S_j + 1, \cdots, M_j , \qquad j = 1, \cdots, n , \tag{17}$$

and

$$\Pr\Big\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j) + y_j \geq 0 \Big\} = 1 , \qquad j = 1, \cdots, n . \tag{39}$$

In general, it appears impossible to obtain solution in closed form to this problem. However, for the special case in which $g(.) = u(.)$ and $Y_{j+1} = 0$, the solution is obtainable without recourse to numeric methods in the case of Models I—III. Since it is not entirely unreasonable for the *shape* of the bequest function to be the same as that of the one-period utility function of consumption and many individuals lack a non-capital income stream, we shall give the solution for that special case.

THEOREM 2: *Let* $a_j, b_j, \beta_{ij}, r_j, F_j$, *and* Y_j *be defined as in Sections* 2 *and* 3. *Moreover, let* $Y_{j+1} = 0$, *let* $g(.) = u(.)$ *and let* $u(c_j)$ *be one of the functions* (28), (29), *or* (30). *Then a solution to* Problem B *exists for* $x_j \geq -Y_j \ (= -y_j/r_j)$ *and is given by, for* $j = 1, \cdots, n$,

$$f_j(x_j) = K_j u(x_j + Y_j) + L_j \ , \tag{78}$$

$$c_j^*(x_j) = N_j(x_j + Y_j) \ , \tag{79}$$

$$z_{ij}^*(x_j) = (1 - N_j)v_{ij}^*(x_j + Y_j) \ , \qquad i = 2, \cdots, M_j \tag{80}$$

$$z_{1j}^*(x_j) = (1 - N_j)(1 - v_j^*)x_j - Y_j(N_j(1 - v_j^*) + v_j^*) \ , \tag{81}$$

where the v_{ij}^* ($v_j^* \equiv \sum_{i=2}^{M_j} v_{ij}^*$) *are the values of* v_{ij} *which maximize* (40) *subject to* (41) *and* (42) *and* k_j *is given by* (50). *The constants* K_j, L_j, *and* N_j *are given by*:

(a) *In the case of* Model I

$$\begin{aligned} K_j &= [1 + (k_j(a_j + b_j(1 + (k_{j+1}(a_{j+1} + b_{j+1}(\cdots \\ &\qquad (1 + (k_n a_n)^{1/(1-\gamma)})^{1-\gamma} \ldots)^{1-\gamma}))^{1/(1-\gamma)})^{1-\gamma}))^{1/(1-\gamma)}]^{1-\gamma} \\ &= A_j^{1-\gamma} \end{aligned} \tag{82}$$

$$L_j = 0 \tag{83}$$

$$N_j = A_j^{-1} \ . \tag{84}$$

(b) *In the case of* Model II

$$\begin{aligned} K_j &= [1 + (-k_j(a_j + b_j(1 + (-k_{j+1}(a_{j+1} + b_{j+1}(\cdots \\ &\qquad (1 + (-k_n a_n)^{1/(\gamma+1)})^{1+\gamma} \ldots)^{1+\gamma}))^{1/(1+\gamma)})^{1+\gamma}))^{1/(1+\gamma)}]^{1+\gamma} \\ &= A_j^{\gamma+1} \end{aligned} \tag{85}$$

$$L_j = 0 \tag{86}$$

$$N_j = A_j^{-1} \ . \tag{87}$$

(c) *In the case of* Model III

$$K_j = 1 + a_j + b_j(1 + a_{j+1} + b_{j+1}(\cdots (1 + a_n))) \ , \tag{88}$$

$$\begin{aligned} L_j &= (K_j - 1)(k_j + \log (K_j - 1)) + b_j(K_{j+1} - 1)(k_{j+1} + \log (K_{j+1} - 1)) \\ &\quad + \cdots + b_j \cdots b_{n-1}(K_n^{-1}))(k_n + \log (K_n^{-1})) - K_j \log K_j \\ &\quad - b_j K_{j+1} \log K_{j+1} - \cdots - b_j \cdots b_{n-1} K_n \log K_n \end{aligned} \tag{89}$$

$$N_j = K_j^{-1} \ . \tag{90}$$

Furthermore, the solution is unique.

The proof is similar to that of Theorem 1 and will therefore be omitted.

It is easily verified that the optimal strategies given by (79), (80), and (81) possess, with minor exceptions, all the properties of strategies (47), (48), and (49). We also note that when (84), (87), and (90) are compared to (53), (56), and (59), the presence of a bequest motive is reflected in the presence of $a_j, \cdots, a_n$ and that the effect of these constants is to decrease $N_j, j = 1, \cdots, n$, i.e., to reduce the fraction of permanent income spent on consumption at all decision points.

7. NO BEQUEST MOTIVE, INSURANCE AVAILABLE

We now return to the case in which the individual has no bequest motive, i.e., in which

$$\delta_j g(x'_{j+1}) = 0 \,, \qquad j = 1, \cdots, n \,. \tag{33}$$

However, we shall now allow the individual to enter into a contract of life insurance $(\bar{t}_j, \cdots, \bar{t}_{n-1})$ at any decision point j where $\bar{t}_j$, as indicated in Section 2, is the contractual premium payable at decision point j. By (25), (23), (33), (15), (16), and (18), the decision problem may now be stated as

PROBLEM C.

$$f_j(x_j) = \max_{c_j, \bar{z}_j} \left\{ u(c_j) + b_j E\left[f_{j+1}\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - \bar{t}_j) + y_j \right) \right] \right\} , \qquad j = 1, \cdots, n \,, \tag{91}$$

where

$$f_{n+1}(x_{n+1}) \equiv 0 \tag{37}$$

$$c_j \geq 0 \,, \qquad j = 1, \cdots, n \,, \tag{38}$$

$$z_{ij} \leq 0 \,, \qquad i = S_j + 1, \cdots, M_j \,, \qquad j = 1, \cdots, n \,, \tag{17}$$

$$\Pr\left\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - \bar{t}_j) + y_j + \bar{t}_j / p_{jj} \geq 0 \right\} = 1 \,, \qquad j = 1, \cdots, n \,. \tag{92}$$

By a proof similar to that of Theorem 1 we obtain

THEOREM 3: *Let $b_j, \beta_{ij}, r_j, F_j, y_j, Y_j, \bar{t}_j$, and $\bar{T}_j$ be defined as in Sections 2 and 3. Moreover, let $\bar{T}_m \leq Y_m$, $m = j, \cdots, n-1$, let $u(c_j)$ be one of the functions (28), (29), or (30), let the v^*_{ij} ($v^*_j \equiv \sum_{i=2}^{M_j} v^*_{ij}$) be the values of v_{ij} which maximize (40) subject to (41) and (42), and let F_k be such that $b(\bar{v}^*_k) > -r_k$ for $k = j, \cdots, m$, where m is the largest k for which $\bar{t}_k < p_{kk}(Y_{k+1} - \bar{T}_{k+1})$. Then for x_j such that*

$$\begin{aligned} x_j &\geq -Y_j + \bar{T}_j \,, && j > m \\ x_j &\geq \max\{\underline{\underline{x}}_j, \underline{\underline{x}}'_j\} \,, && j \leq m \end{aligned} \tag{93}$$

where

$$\begin{aligned} \underline{\underline{x}}_k &= \frac{\bar{\underline{\underline{x}}}_{k+1} + Y_{k+1} - \bar{T}_{k+1}}{(1 - N_k)(b(\bar{v}^*_k) + r_k)} - Y_k + \bar{T}_k \,, && k = j, \cdots, m \,, \\ \underline{\underline{x}}'_k &= \frac{Y_{k+1} - \bar{T}_{k+1} - \bar{t}_k / p_{kk}}{(1 - N_k)(b(\bar{v}^*_k) + r_k)} - Y_k + \bar{T}_k \,, && k = j, \cdots, m \,, \\ \bar{\underline{\underline{x}}}_k &= \max\{\underline{\underline{x}}_k, \underline{\underline{x}}'_k\} \,, && k = j+1, \cdots, m \,, \\ \bar{\underline{\underline{x}}}_{m+1} &= -Y_{m+1} + T_{m+1} \,, \end{aligned} \tag{94}$$

the solution to Problem C *is given by, for* $\qquad j = 1, \cdots, n \,,$

(95) $$f_j(x_j) = K_j u(x_j + Y_j - \bar{T}_j) + L_j$$

(96) $$c_j^*(x_j) = N_j(x_j + Y_j - \bar{T}_j)$$

(97) $$z_{ij}^*(x_j) = (1 - N_j)v_{ij}^*(x_j + Y_j - \bar{T}_j)\,, \qquad i = 2, \cdots, M_j\,,$$

(98) $$z_{1j}^*(x_j) = (1 - N_j)(1 - v_j^*)x_j - (Y_j - \bar{T}_j)(N_j(1 - v_j^*) + v_j^*) - \bar{t}_j$$

where k_j *is given by* (50), *and the constants* K_j, L_j, *and* N_j *are given by* (51), (52), *and* (53) *in the case of* Model I, (54), (55), *and* (56) *in the case of* Model II, *and* (57), (58), *and* (59) *in the case of* Model III. *Furthermore, the solution is unique.*

As in Problem A, we note that by the Corollary, it is possible for $b(\bar{v}_j^*) + r_j$ to assume the value 0 only in the case of Model I. Thus, we observe from (94) that when in Model I $b(v_k^*) + r_k = 0$ in some period $k \geq j$, and in addition

(99) $$\bar{t}_k / p_{kk} < Y_{k+1} - \bar{T}_{k+1}\,,$$

there does not exist any $x_j < \infty, j < n$, for which the solution to Problem C has the form given by Theorem 3.

Again, all the properties of strategies (47), (48), and (49) are possessed by the optimal strategies (96), (97), and (98).

Differentiating (95) with respect to $\bar{T}_j$ we obtain

(100) $$\frac{\partial f_j}{\partial \bar{T}_j} = -K_j u'(x_j + Y_j - \bar{T}_j) < 0\,.$$

Thus, whenever x_j is such that (93) holds when $\bar{T}_j = 0$, i.e., (43b) holds, we find that the individual is better off without insurance than with insurance. In this case, the optimal insurance contract $(\bar{t}_j^*, \cdots, \bar{t}_{n-1}^*)$ is consequently given by

$$(\bar{t}_j^*, \cdots, \bar{t}_{n-1}^*) = (0, \cdots, 0)\,.$$

The only possible justification for buying insurance when there is no bequest motive, then, is for the purpose of satisfying constraint (92). In view of (100), (93), and (99), it is also clear that it never pays to insure more than $Y_{k+1} - \bar{T}'_{k+1}$ in any period k, i.e., that

(101) $$(0, \cdots, 0) \leq (\bar{t}_j^*, \cdots, \bar{t}_{n-1}^*) \leq (\bar{t}'_j, \cdots, \bar{t}'_{n-1})$$

where $\bar{t}'_k$ is obtained by backward recursion on k from

(102) $$\bar{t}'_k \equiv (Y_{k+1} - \bar{T}'_{k+1})p_{kk} = \left(Y_{k+1} - \bar{t}'_{k+1} - \cdots - \frac{\bar{t}'_{n-1}}{r_{k+1} \cdots r_{n-2}}\right)p_{kk}\,, \qquad k = j, \cdots, n-1\,.$$

8. BEQUEST MOTIVE, INSURANCE AVAILABLE

We now turn to the case in which both the bequest motive is present and insurance contracts may be purchased. By (25), (15), (16), and (18) we then obtain

PROBLEM D.

$$f_j(x_j) = \max_{c_j, z_j} \Big\{ u(c_j) + E\Big[a_j g\Big(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - \bar{t}_j) + y_j + \bar{t}_j / p_{jj} \Big) + b_j f_{j+1}\Big(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - \bar{t}_j) + y_j \Big) \Big] \Big\} \quad j = 1, \cdots, n, \tag{103}$$

where

$$f_{n+1}(x_{n+1}) \equiv 0 \tag{37}$$

$$c_j \geq 0, \qquad j = 1, \cdots, n, \tag{38}$$

$$z_{ij} \geq 0, \quad i = S_j + 1, \cdots, M_j, \qquad j = 1, \cdots, n, \tag{17}$$

$$\Pr\Big\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - \bar{t}_j) + \bar{t}_j / p_{jj} \geq 0 \Big\} = 1 \qquad j = 1, \cdots, n. \tag{92}$$

As in the case of Problem C, this problem does not appear to lend itself to solution in closed form in the general case. However, when $g(.) = u(.)$ and $\bar{t}_k = \bar{t}'_k$, $k = j, \cdots, n - 1$, the structure of the solution to Problems A-C can again be shown to hold. Specifically, we obtain the following result:

THEOREM 4: *Let $a_j, b_j, \beta_{ij}, r_j, F_j, y_j, Y_j, \bar{t}_j$, and $\bar{T}_j$ be defined as in Sections 2 and 3. Moreover, let $\bar{t}_m = \bar{t}'_m$, $m = j, \cdots, n - 1$, let $g(.) = u(.)$, and let $u(c_j)$ be one of the functions* (28), (29), *or* (30). *Then a solution to* Problem D *exists for $x_j \geq -(Y_j - \bar{T}'_j)$ and is given by, for $j = 1, \cdots, n$,*

$$f_j(x_j) = K_j u(x_j + Y_j - \bar{T}'_j) + L_j \tag{104}$$

$$c_j^*(x_j) = N_j(x_j + Y_j - \bar{T}'_j) \tag{105}$$

$$z_{ij}^*(x_j) = (1 - N_j) v_{ij}^*(x_j + Y_j - \bar{T}'_j), \qquad i = 2, \cdots, M_j, \tag{106}$$

$$z_{1j}^*(x_j) = (1 - N_j)(1 - v_j^*) x_j - (Y_j - \bar{T}'_j)(N_j(1 - v_j^*) + v_j^*) - \bar{t}'_j \tag{107}$$

where the v_{ij}^* ($v_j^* \equiv \sum_{i=2}^{M_j} v_{ij}^*$) are the values of v_{ij} which maximize (40) *subject to* (41) *and* (42), *k_j is given by* (50), *and the constants K_j, L_j, and N_j are given by* (82), (83), *and* (84) *in the case of* Model I, (85), (86), *and* (87) *in the case of* Model II, *and* (88), (89), *and* (90) *in the case of* Model III. *Furthermore, the solution is unique.*

Again, the proof is similar to those of the preceding theorems and is therefore omitted. As before, the optimal strategies (105), (106), and (107) possess the same properties as (79), (80), and (81).

9. THE COMPLETE PROBLEM

So far we have treated insurance, when available, as an exogenous variable; both in Problem C and in Problem D an insurance contract $(\bar{t}_j, \cdots, \bar{t}_{n-1})$ was consumated by the individual on or before the current decision point. It would seem more appropriate, however, for the amount of insurance to be purchased in any period to be a decision variable along with the amount to consume and the amounts to invest in the various risky opportunities, as well as the amount to borrow or to lend. After all, as is clear from Problems C and D, the individual is in a position to influence the total expected utility obtainable over time by his choice of insurance coverage over his remaining life-time.

On the basis of (25), (15), (16), and (18) we obtain when t_j is a decision variable

PROBLEM E.

$$f_j(x_j) = \max_{c_j, \bar{z}_j, t_j} \Big\{ u(c_j) + E\Big[a_j g\Big(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - t_j) + y_j + t_j/p_{jj} \Big) + b_j f_{j+1}\Big(\sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - t_j) + y_j \Big) \Big] \Big\} \quad j = 1, \cdots, n, \tag{108}$$

where

$$f_{n+1}(x_{n+1}) \equiv 0 \tag{37}$$

$$c_j \geq 0, \qquad j = 1, \cdots, n, \tag{38}$$

$$z_{ij} \geq 0, \quad i = S_j + 1, \cdots, M_j, \qquad j = 1, \cdots, n, \tag{17}$$

$$t_j \geq 0 \qquad j = 1, \cdots, n-1, \tag{109}$$

$$t_n = 0 \tag{110}$$

$$\Pr\Big\{ \sum_{i=2}^{M_j} (\beta_{ij} - r_j) z_{ij} + r_j(x_j - c_j - t_j) + y_j + t_j/p_{jj} \geq 0 \Big\} = 1, \quad j = 1, \cdots, n. \tag{111}$$

As in the previous cases, Problem E seems susceptible to solution by non-numeric methods only when the utility functions $u(.)$ and $g(.)$ are of rather special form. The case when there is no bequest motive and $u(c)$ is given by one of Models I-III was discussed in Section 7, where the conditions under which $t_k^*(x_k) = 0, k = j, \cdots, n-1$, were given. In addition, (101) implies that it would never be optimal to take out insurance in an amount greater than the present value of the non-capital income stream in any period. Thus, an individual with no non-capital income stream and no bequest motive would not buy any life insurance offered at a "fair" rate under an optimal strategy. The solution to Problem E when there is no bequest motive, $Y_{j+1} > 0$, and x_j is small enough for constraint (111) to be binding does not appear to be obtainable by non-numeric methods (see Section 7).

Turning to Problem E when $g(.) = u(.)$, the solution is even more elusive; in no case does it appear to exist in closed form. Let us therefore seek the conditions under which the strategy $t_j(x_j) = \bar{t}'_j, j = 1, \cdots, n-1$, in Problem D would be optimal in Problem E. Differentiating (108) with respect to t_j and inserting (104)-(107) on the right-hand side we obtain

$$(112) \qquad \frac{\partial f_j}{\partial t_j} = [a_j(-r_j + 1/p_{jj}) - b_j K_{j+1} r_j] E[u'(x_{j+1} + Y_{j+1} - \bar{T}'_{j+1})] .$$

In order for (112) to be zero, which is sufficient for a maximum whenever (39) does not bind by the strict concavity of u, we must have

$$(113) \qquad a_j(-r_j + 1/p_{jj}) - b_j K_{j+1} r_j = 0$$

since the last factor in (112) is always positive by the monotonicity of $u(c)$. (113) now gives

$$(114) \qquad a_k = b_k K_{k+1} r_k / (1/p_{kk} - r_k) , \qquad k = j, \cdots, n-1 .$$

But, since the numerator of (114) is positive and since $a_k > 0$ by (23),

$$(115) \qquad p_{kk} r_k < 1 , \qquad k = j, \cdots, n-1$$

must also hold in order that

$$(116) \qquad t_j^*(x_j) = \bar{t}'_j .$$

This gives

THEOREM 5. *Let* $a_j, b_j, \beta_{jj}, r_j, F_j, y_j, Y_j, t_j$, *and* T_j *be defined as in Sections* 2 *and* 3. *Moreover, let* $g(.) = u(.)$ *and let* $u(c_j)$ *be one of the functions* (28), (29), *or* (30). *Then the solution to* Problem E *is given, whenever* $p_{kk} r_k < 1$ *and* a_k *is given by* (114) *for* $k = j, \cdots, n-1$, *by the solution to* Problem D, *as stated in* Theorem 4, *with the addition*

$$(116) \qquad t_j^*(x_j) = \bar{t}'_j$$

inserted after equation (107).

10. IMPLICATIONS WITH RESPECT TO THE THEORY OF THE FIRM

In the case of a known horizon, it was shown in an earlier paper [6] that, starting with a collection of heterogeneous individuals, each of whom is bent on maximizing (his own) utility from consumption over time in accordance with one of Models I-III, there exists a basis for the formation of firms by *subcollections* of individuals, where each subcollection in turn is heterogeneous with respect to wealth, non-capital income, and impatience to consume. Each firm so formed is uniquely characterizable by the pair $(q^*, \langle F_j \rangle)$ and has a well-defined (unique) objective function, which may be interpreted as imputing a precise meaning to the term "profit maximization" under risk and with respect to time. The capital structure of each firm turns out to be unimportant except when it has limited liability, in which case there is an upper limit on the firm's feasible debt-equity ratio. Since the "long-run objective" is achieved by repeated performance of the short-run maximization process, it is also apparent that while the owners(stockholders) all have finite life-times, the firm itself may have an infinite life.

The preceding results derive from the fact that, under a known horizon, the optimal *mix* of risky investments is a function only of q^*, F_j, and r_j, where r_j presumably is the same for everyone. Turning to (48), (80), (97), and (106), it is readily verified that the optimal mix of risky investments, in the cases encompassed by Theorems 1–5, also depends only on q^*, F_j, and r_j. As a result, it can easily be shown that the preceding results with respect to the formation and operation of firms also hold under the conditions of Theorems 1–5. Consequently, the owners (stockholders) of firm $(q^*, \langle F_j \rangle)$ may not only differ in wealth, non-capital income, impatience, and age but may have random life-times and may differ in the strength of their bequest motives as well as in their insurance contracts.

11. ON THE SUPPLY OF INSURANCE

So far we have assumed that the individual can only buy, and not sell, insurance on his own life. We shall now justify this assumption by showing that other individuals can be expected to be willing sellers, but not buyers, of insurance on his life.

Let us assume that in addition to the regular M_j investment opportunities available in period j, the individual is given the opportunity to buy insurance on the lives of individuals $M_j + 1, \cdots, M_j'$ at the j-th decision point. The return per unit of premium invested in opportunity $i, i = M_j + 1, \cdots, M_j'$, will, as before, be denoted β_{ij}. In accordance with the preceding, β_{ij} will then assume, at the end of the period, the value 0 with probability $1 - p_{jj}^i$ and the value $1/p_{jj}^i$ with probability $p_{jj}^i, i = M_j + 1, \cdots, M_j'$ where p_{jj}^i is the probability that individual i will pass away in period j (given that he is alive at decision point j). Thus, $E[\beta_{ij}] = 1, i = M_j + 1, \cdots, M_j'$. Moreover, we assumed in Section 2 that the $\beta_{ij}, i = M_j + 1, \cdots, M_j'$, are statistically independent. Thus we may write, upon differentiating (40),

$$\begin{aligned}\frac{\partial h_j}{\partial v_{ij}}(v_{ij} = 0) &= E\left[u'\left(\sum_{k=2}^{M_j'} (\beta_{kj} - r_j)v_{kj} + r_j\right)\right]E[\beta_{ij} - r_j] \\ &\qquad\qquad i = M_j + 1, \cdots, M_j', \text{ all } j \\ &= (1 - r_j)E\left[u'\left(\sum_{k=2}^{M_j'} (\beta_{kj} - r_j)v_{kj} + r_j\right)\right] \\ &< 0\end{aligned}$$

since $r_j > 1$ and the second factor is positive for all feasible $\bar{v}_j$ by the monotonicity of u. As a result, since h_j is strictly concave in $\bar{v}_j$ [8], $v_{ij}^* \le 0$, $i = M_j + 1, \cdots, M_j'$ and inequality would hold whenever the solution $\bar{v}_k^*$ to (40) subject to (41) and (42) is interior with respect to (41). As shown by the Corollary, this is always the case in Models II and III since they have no lower bound on u. Since $z_{ij}^*(x_j) \le 0$ whenever $v_{ij}^* \le 0$, an individual would thus never buy insurance on the life of another, thereby making it impossible for anyone to *sell* insurance on his own life. However, short sales, or simply sales, of insurance on the lives of others generally enable, as we have seen, the individual to increase his own utility since an increase in k_j produces an increase in $f_j(x_j)$. Thus, whenever an individual decides to buy insurance on his own life at the postulated rate, he will always find willing suppliers of that insurance among those of his fellow men who obey one of Models I-III.

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