ECONOMETRICA

VOLUME 38 September, 1970 NUMBER 5

OPTIMAL INVESTMENT AND CONSUMPTION STRATEGIES UNDER RISK FOR A CLASS OF UTILITY FUNCTIONS[1]

BY NILS H. HAKANSSON[2]

This paper develops a sequential model of the individual's economic decision problem under risk. On the basis of this model, optimal consumption, investment, and borrowing-lending strategies are obtained in closed form for a class of utility functions. For a subset of this class the optimal consumption strategy satisfies the permanent income hypothesis precisely. The optimal investment strategies have the property that the optimal mix of risky investments is independent of wealth, noncapital income, age, and impatience to consume. Necessary and sufficient conditions for long-run capital growth are also given.

1. INTRODUCTION AND SUMMARY

THIS PAPER presents a normative model of the individual's economic decision problem under risk. On the basis of this model, optimal consumption, investment, and borrowing-lending strategies are obtained in closed form for a class of utility functions. The model itself may be viewed as a formalization of Irving Fisher's model of the individual under risk, as presented in *The Theory of Interest* [4]; at the same time, it represents a generalization of Phelps' model of personal saving [10].

The various components of the decision problem are developed and assembled into a formal model in Section 2. The objective of the individual is postulated to be the maximization of expected utility from consumption over time. His resources are assumed to consist of an initial capital position (which may be negative) and a noncapital income stream which is known with certainty. The individual faces both financial opportunities (borrowing and lending) and an arbitrary number of productive investment opportunities. The returns from the productive opportunities are assumed to be random variables, whose probability distributions satisfy the "no-easy-money condition." The fundamental characteristic of the approach taken is that the portfolio composition decision, the financing decision, and the consumption decision are all analyzed simultaneously in *one* model. The vehicle of analysis is discrete-time dynamic programming.

In Section 3, optimal strategies are derived for the class of utility functions $\sum_{j=1}^{\infty} \alpha^{j-1}u(c_j)$, $0 < \alpha < 1$, where c_j is the amount of consumption in period j, such that either the relative risk aversion index, $-cu''(c)/u'(c)$, or the absolute risk

[1] This paper was presented at the winter meeting of the Econometric Society, San Francisco, California, December, 1966.

[2] This article is based on my dissertation which was submitted to the Graduate School of Business Administration of the University of California, Los Angeles, in June, 1966. I am greatly indebted to Professors George W. Brown (committee chairman), Jacob Marschak, and Jacques Drèze for many valuable suggestions and comments and to Professors Jack Hirshleifer, Leo Breiman, James Jackson, and Fred Weston for constructive criticisms. I am also grateful to the Ford Foundation for financial support over a three-year period.

aversion index, $-u''(c)/u'(c)$, is a positive constant for all $c \geqslant 0$, i.e., $u(c) = c^\gamma$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma > 0$, $u(c) = \log c$, and $u(c) = -e^{-\gamma c}$, $\gamma > 0$.

Section 4 is devoted to a discussion of the properties of the optimal consumption strategies, which turn out to be linear and increasing in wealth and in the present value of the noncapital income stream. In three of the four models studied, the optimal consumption strategies precisely satisfy the properties specified by the consumption hypotheses of Modigliani and Brumberg [**9**] and of Friedman [**5**]. The effects of changes in impatience and in risk aversion on the optimal amount to consume are found to coincide with one's expectations. In response to changes in the "favorableness" of the investment opportunities, however, the four models exhibit an exceptionally diverse pattern with respect to consumption behavior.

The optimal investment strategies have the property that the optimal mix of risky (productive) investments in each model is *independent* of the individual's wealth, noncapital income stream, and impatience to consume. It is shown in Section 5 that the optimal mix depends in each case only on the probability distributions of the returns, the interest rate, and the individual's one-period utility function of consumption. This section also discusses the properties of the optimal lending and borrowing strategies, which are linear in wealth. Three of the models always call for borrowing when the individual is poor while the fourth model always calls for lending when he is sufficiently rich. The effect of differing borrowing and lending rates is also examined.

Necessary and sufficient conditions for capital growth are derived in Section 6. It is found that when the one-period utility function of consumption is logarithmic, the individual will always invest the capital available after the allotment to current consumption so as to maximize the expected growth rate of capital plus the present value of the noncapital income stream. Finally, Section 7 indicates how the preceding results are modified in the nonstationary case and under a finite horizon.

2. THE MODEL

In this section we shall combine the building blocks discussed in the previous section into a formal model. The following notation and assumptions will be employed:

c_j: amount of consumption in period j, where $c_j \geqslant 0$ (decision variable).

$U(c_1, c_2, c_3, \ldots)$: the utility function, defined over all possible consumption programs $(c_1, c_2, c_3, \ldots)$. The class of functions to be considered is that of the form

$$U(c_1, c_2, c_3, \ldots) = u(c_1) + \alpha U(c_2, c_3, c_4, \ldots) = \sum_{j=1}^{\infty} \alpha^{j-1} u(c_j), \quad 0 < \alpha < 1. \tag{1}$$

It is assumed that $u(c)$ is monotone increasing, twice differentiable, and strictly concave for $c \geqslant 0$. The objective in each case is to maximize $E[U(c_1, c_2, \ldots)]$, i.e., the expected utility derived from consumption over time.[3]

x_j: amount of capital (debt) on hand at decision point j (the beginning of the jth period) (state variable).

y: income received from noncapital sources at the *end* of each period, where $0 \leqslant y < \infty$.

[3] While we make use of the expected utility theorem, we assume that the von Neumann-Morgenstern postulates [**12**] have been modified in such a way as to permit unbounded utility functions.

M : the number of available investment opportunities.

S : the subset of investment opportunities which it is possible to sell short.

z_{ij}: amount invested in opportunity i, $i = 1, \ldots, M$, at the beginning of the jth period (decision variable).

$r - 1$: rate of interest, where $r > 1$.

β_i : transformation of each unit of capital invested in opportunity i in any period j (random variable); that is, if we invest an amount θ in i at the beginning of a period, we will obtain $\beta_i\theta$ at the end of that period (stochastically constant returns to scale, no transaction costs or taxes). The joint distribution functions of the β_i, $i = 1, \ldots, M$, are assumed to be known and independent with respect to time j. The $\{\beta_i\}$ have the following properties:

$$\beta_1 = r, \tag{2}$$

$$0 \leqslant \beta_i < \infty \qquad (i = 2, \ldots, M), \tag{3}$$

$$\Pr\left\{\sum_{i=2}^{M} (\beta_i - r)\theta_i < 0\right\} > 0, \tag{4}$$

for all finite θ_i such that $\theta_i \geqslant 0$ for all $i \notin S$ and $\theta_i \neq 0$ for at least one i.

$f_j(x_j)$: expected utility obtainable from consumption over all future time, evaluated at decision point j, when capital at that point is x_j and an optimal strategy is followed with respect to consumption and investment.

Y: present value at any decision point of the noncapital income stream capitalized at the rate of interest, i.e., $Y = y/(r - 1)$.

$\bar{v} \equiv (v_2, \ldots, v_M)$: a vector of real numbers.

$h(\bar{v}) \equiv E\left[u\left(\sum_{i=2}^{M} (\beta_i - r)v_i + r\right)\right]$.

k : maximum of $h(\bar{v})$ subject to (27) and (28) (see (26)).

$\bar{v}^*$: vector $\bar{v}$ which gives maximum k of $h(\bar{v})$ (see (26)).

$v^* \equiv \sum_{i=2}^{M} v_i^*$.

$c^*(x)$: an optimal consumption strategy.

$z_1^*(x)$: an optimal lending strategy.

$z_i^*(x)$: an optimal investment strategy for opportunity i, $i = 2, \ldots, M$.

$s_j \equiv x_j + Y$.

The limitations of utility functions of the form (1) are well known and need not be elaborated here. Condition (4) will be referred to as the "no-easy-money condition." In essence, this condition states (i) that no combination of productive investment opportunities exists which provides, with probability 1, a return at least as high as the (borrowing) rate of interest; (ii) that no combination of short sales exists in which the probability is zero that a loss will exceed the (lending) rate of interest; (iii) that no combination of productive investments made from the proceeds of any short sale can guarantee against loss. For these reasons, (4) may be viewed as a condition that the prices of the various assets in the market must satisfy in equilibrium.

Consumption and investment decisions are assumed to be made at the beginning of each period. The amount allocated to consumption is assumed to be spent immediately or, if spent gradually over the period, to be set aside in a nonearning account. We also assume that any debt incurred by the individual must at all times be fully secured, i.e., that the individual must be solvent at each decision point. In view of the "no-easy-money condition" (4), this implies that his (net) debt cannot exceed the present value, on the basis of the (borrowing) rate of interest, of his noncapital income stream at the end of any period.

We shall now identify the relation which determines the amount of capital (debt) on hand at each decision point in terms of the amount on hand at the previous decision point. This leads to the difference equation

$$x_{j+1} = rz_{1j} + \sum_{i=2}^{M} \beta_i z_{ij} + y \qquad (j = 1,2,\ldots) \tag{5}$$

where

$$\sum_{i=1}^{M} z_{ij} = x_j - c_j \qquad (j = 1,2,\ldots). \tag{6}$$

The first term of (5) represents the payment of the debt or the proceeds from savings, the second term the proceeds from productive investments, and the third term the noncapital income received. Combining (5) and (6) we obtain

$$x_{j+1} = \sum_{i=2}^{M} (\beta_1 - r)z_{ij} + r(x_j - c_j) + y \qquad (j = 1,2,\ldots). \tag{7}$$

This is the difference equation, then, which governs the process we are about to study.

The definition of $f_j(x_j)$ may formally be written

$$f_j(x_j) \equiv \max E[U(c_j, c_{j+1}, c_{j+2}, \ldots)]|x_j. \tag{8}$$

From (1) we obtain, by the principle of optimality,[4] for all j,

$$f_j(x_j) = \max E[u(c_j) + \alpha\{\max E[U(c_{j+1}, c_{j+2}, \ldots)]|x_{j+1}\}]|x_j, \tag{9}$$

since we have assumed the $\{\beta_i\}$ to be independently distributed with respect to time j. By (8), (9) reduces to

$$f_j(x_j) = \max\{u(c_j) + \alpha E[f_{j+1}(x_{j+1})]\}, \quad \text{all } j. \tag{10}$$

Since by our assumptions we are faced with exactly the same problem at decision point $j+1$ as when we are at decision point j, the time subscript may be dropped. Using (7), (10) then becomes

$$f(x) = \max_{c,\{z_i\}} \left\{ u(c) + \alpha E\left[f\left(\sum_{i=2}^{M} (\beta_i - r)z_i + r(x - c) + y \right) \right] \right\} \tag{11}$$

subject to

$$c \geq 0, \tag{12}$$

$$z_i \geq 0, \quad i \notin S, \tag{13}$$

and

$$\Pr\left\{ \sum_{i=2}^{M} (\beta_i - r)z_i + r(x - c) + y \geq -Y \right\} = 1 \tag{14}$$

at each decision point. Expression (14), of course, represents the solvency constraint.

[4] The principle of optimality states that an optimal strategy has the property that whatever the initial state and the initial decision, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision [2, p. 83].

For comparison, the model studied by Phelps [10] is given by the functional equation

$$f(x) = \max_{0 \leq c \leq x} \{u(c) + \alpha E[f(\beta(x - c) + y)]\}. \tag{15}$$

In this model, all capital not currently consumed obeys the transformation β, which is identically and independently distributed in each period. Since the amount invested, $x - c$, is determined once c is known, (15) has only one decision variable (c).[5]

Since x represents capital, $f(x)$ is clearly the utility of money at any decision point j. Instead of being assumed, as is generally the case, the utility function of money has in this model been induced from inputs which are more basic than the preferences for money itself. As (11) shows, $f(x)$ depends on the individual's preferences with respect to consumption, his noncapital income stream, the interest rate, and the available investment opportunities and their riskiness.

3. THE MAIN THEOREMS

We shall now give the solution to (11) for the class of one-period utility functions

$$u(c) = \frac{1}{\gamma} c^{\gamma}, \qquad 0 < \gamma < 1 \qquad \text{(Model I)}; \tag{16}$$

$$u(c) = \frac{1}{\gamma} c^{\gamma}, \qquad \gamma < 0 \qquad \text{(Model II)}; \tag{17}$$

$$u(c) = \log c, \qquad \text{(Model III)}; \tag{18}$$

$$u(c) = -e^{-\gamma c}, \qquad \gamma > 0 \qquad \text{(Model IV)}. \tag{19}$$

[5] Phelps gives the solution to (15) for the utility functions $u(c) = c^{\gamma}$, $0 < \gamma < 1$, $u(c) = -c^{-\gamma}$, $\gamma < 0$, and for $u(c) = \log c$ when $y = 0$. Unfortunately, this solution is incorrect in the general case, i.e., whenever $y > 0$ *and* the distribution of β is nondegenerate. For example, when $u(c) = -c^{-\gamma}$, the solution is asserted to be, letting $\bar{\beta} \equiv E[\beta^{-\gamma}]$,

$$f(x) = -\left[\frac{(\alpha\bar{\beta})^{-1/(\gamma+1)}}{(\alpha\bar{\beta})^{-1/(\gamma+1)} - 1}\right]^{\gamma+1}\left[x + \frac{y}{\bar{\beta}^{-1/\gamma} - 1}\right]^{-\gamma}, \tag{15a}$$

$$c(x) = [1 - (\alpha\bar{\beta})^{1/(\gamma+1)}]\left[x + \frac{y}{\bar{\beta}^{-1/\gamma} - 1}\right], \tag{15b}$$

whenever $\alpha\bar{\beta} < 1$. But for this to be a solution, it would be necessary that one be able to write

$$E[(\beta(x - c) + y)^{-\gamma}] = E[\beta^{-\gamma}]\left[x - c + \frac{y}{E[\beta^{-\gamma}]^{-1/\gamma}}\right]^{-\gamma} \tag{15c}$$

which is clearly impossible unless the distribution of β is degenerate or $y = 0$ or both. The right side of (15c) may, of course, be regarded as a first-order approximation of the left side when the variance of β is small, but this negates the presence of uncertainty. In fact, the preceding solution holds even under certainty only when $\alpha\beta \geqslant 1$ and $x \geqslant [(\alpha\bar{\beta})^{-1/(\gamma+1)} - 1]y/(\bar{\beta} - 1)$, i.e., when $c(x)$ is less than or equal to x in *all* future periods.

It appears that an analytic solution to (15) does not exist when $y > 0$ and the distribution of β is nondegenerate. It is ironic, therefore, that when one generalizes Phelps' problem by introducing the possibility of *choice* among risky investment opportunities *and* the opportunity to borrow and lend (see (11)), an analytic solution does exist (as will be shown). It is the second of these generalizations which guarantees the solution in closed form.

Pratt [11] notes that (16)–(18) are the only monotone increasing and strictly concave utility functions for which the relative risk aversion index

$$(20)\qquad q^*(c) \equiv -\frac{u''(c)c}{u'(c)}$$

is a positive constant and that (19) is the only monotone increasing and strictly concave utility function for which the absolute risk aversion index

$$(21)\qquad q(c) \equiv -\frac{u''(c)}{u'(c)}$$

is a positive constant.[6]

THEOREM 1: *Let $u(c)$, α, y, r, $\{\beta_i\}$, and Y be defined as in Section 2. Then, whenever $u(c)$ is one of the functions (16)–(18) and $k\gamma < 1/\alpha$ in Model I, a solution to (11) subject to (12)–(14) exists for $x \geqslant -Y$ and is given by*

$$(22)\qquad f(x) = Au(x+Y) + C,$$

$$(23)\qquad c^*(x) = B(x+Y),$$

$$(24)\qquad z_1^*(x) = (1-B)(1-v^*)(x+Y) - Y,$$

$$(25)\qquad z_i^*(x) = (1-B)v_i^*(x+Y) \qquad (i = 2,\ldots,M)$$

where the constants v_i^* ($v^* \equiv \sum_{i=2}^M v_i^*$) and k are given by

$$(26)\qquad k \equiv E\left[u\left(\sum_{i=2}^M (\beta_i - r)v_i^* + r\right)\right] = \max_{\{v_i\}} E\left[u\left(\sum_{i=2}^M (\beta_i - r)v_i + r\right)\right],$$

subject to

$$(27)\qquad v_i \geqslant 0, \quad i \notin S,$$

and

$$(28)\qquad Pr\left\{\sum_{i=2}^M (\beta_i - r)v_i + r \geqslant 0\right\} = 1,$$

and the constants A, B, and C are given by

(i) *in the case of Models I–II,*

$$(29)\qquad \begin{aligned} A &= (1-(\alpha k\gamma)^{1/(1-\gamma)})^{\gamma-1}, \\ B &= 1-(\alpha k\gamma)^{1/(1-\gamma)}, \\ C &= 0; \end{aligned}$$

[6] The underlying mathematical reason why solutions are obtained in closed form (Theorems 1 and 2) for the utility functions (16)–(19) is that these functions are also the only (monotone increasing and strictly concave utility function) solutions (see [8]) to the functional equations $u(xy) = v(x)w(y)$, $u(xy) = v(x) + w(y)$, $u(x+y) = v(x)w(y)$, and $u(x+y) = v(x) + w(y)$, which are known as the generalized Cauchy equations [1, p. 141].

(ii) *in the case of Model III,*

$$A = \frac{1}{1-\alpha},$$

$$B = 1 - \alpha, \tag{30}$$

$$C = \frac{1}{1-\alpha}\log(1-\alpha) + \frac{\alpha\log\alpha}{(1-\alpha)^2} + \frac{\alpha k}{(1-\alpha)^2}.$$

Furthermore, the solution is unique.

In proving this theorem, we shall make use of the following lemma and corollaries.

LEMMA: *Let $u(c)$, $\{\beta_i\}$, and r be defined as in Section 2 and let $\bar{v} \equiv (v_2, \ldots, v_M)$ be a vector of real numbers. Then the function*

$$h(v_2, v_3, \ldots, v_M) \equiv E\left[u\left(\sum_{i=2}^{M}(\beta_i - r)v_i + r\right)\right] \tag{31}$$

subject to the constraints

$$v_i \geqslant 0, \quad i \notin S, \tag{27}$$

and

$$Pr\left\{\sum_{i=2}^{M}(\beta_i - r)v_i + r \geqslant 0\right\} = 1, \tag{28}$$

has a maximum and the maximizing v_i ($\equiv v_i^*$) are finite and unique.

PROOF: Let D be the $(M-1)$-dimensional space defined by the set of points $\bar{v}$ which satisfy (27) and (28). We shall first prove that the set D is nonempty, closed, bounded, and convex, and that h is strictly concave on D.[7]

The nonemptiness of D follows trivially from the observation that $\bar{v}^0 \equiv (0, 0, \ldots, 0)$ is a member of D. By the boundedness of the β_i's and of r ((2) and (3)), there exists a neighborhood of $\bar{v}^0$ in relation to D. That is, there is a neighborhood of points $\bar{v}'$ such that

$$\Pr\left\{\sum_{i=2}^{M}(\beta_i - r)v_i' + r \geqslant 0\right\} = 1$$

where $v_i' \geqslant 0$ for all $i \notin S$.

Now consider the point $\bar{v}^\lambda \equiv \bar{v}^0 + \lambda\bar{v}' = \lambda\bar{v}'$ where $\lambda \geqslant 0$ and $\bar{v}'$ is one of the points in this neighborhood. Let $b(\bar{v})$ be the greatest lower bound on b such that

$$\Pr\left\{\sum_{i=2}^{M}(\beta_i - r)v_i < b\right\} > 0.$$

[7] The author gratefully acknowledges a debt to Professor George W. Brown for several valuable suggestions concerning the proof of the closure and the boundedness of D.

By the "no-easy-money condition" (4), $b(\bar{v}') \geqslant -r$ for $\bar{v}' \in D$, $b(\bar{v}^0) = 0$, and $b(\bar{v}) < 0$ for all $\bar{v} \neq \bar{v}^0$. Applying the "no-easy-money condition" with respect to the point $\bar{v}^\lambda$ and using the inequality

$$\Pr\left\{\sum_{i=2}^{M} (\beta_i - r)\lambda v_i < \lambda b\right\} > 0,$$

we obtain that $\lambda b(\bar{v}') = b(\lambda\bar{v}')$. But when $\lambda b(\bar{v}') < -r$, or $\lambda > -r/b(\bar{v}')$, the point $\bar{v}^\lambda$ cannot lie in D since $\lambda > -r/b(\bar{v}')$ implies that

$$\Pr\left\{\sum_{i=2}^{M} (\beta_i - r)\lambda v_i' + r \geqslant 0\right\} < 1.$$

Thus, $\lambda_0 \equiv -r/b(\bar{v}')$ is the greatest lower bound on λ such that $\bar{v}^\lambda \notin D$. Since $\lambda_0 b(\bar{v}') = -r$, $\bar{v}^{\lambda_0} \in D$ and is in fact the point farthest from $\bar{v}^0$ lying on the line through $\bar{v}^0$ and $\bar{v}'$ and belonging to D.

We shall only sketch the remainder of the proof establishing the closure and boundedness of D. Let $\bar{v} \neq \bar{v}^0$ be the limit of a sequence of points $\bar{v}^{(n)} \in D$. Since each point in the sequence belongs to D, $b(\bar{v}^{(n)}) \geqslant -r$ for all n. It can now be shown, by utilizing the fact that $\Sigma_{i=2}^{M} (\beta_i - r)\bar{v}_i$ is continuous at any $\bar{v} \neq \bar{v}^0$, uniformly with respect to the β_i's on any bounded set, that $\overline{\lim}_{n\to\infty} b(\bar{v}^{(n)}) \leqslant b(\bar{v})$, which implies that $\bar{v} \in D$. Consequently, D must be closed.

The boundedness of D is established as follows. Let S_R be the set of points $\bar{v}$ such that $|\bar{v}| = R > 0$. S_R is then clearly both closed and bounded. If $D' \equiv D \cap S_R$ is empty, the boundedness of D follows immediately. Let us therefore assume that D' is nonempty; in this case D' is also bounded and closed since D is closed and S_R is bounded and closed. If $\bar{v}$ is a limit point of the sequence $\langle \bar{v}^{(n)} \rangle$ such that $\bar{v}^{(n)} \in D'$, we must have that $\bar{v} \in D'$ since D' is closed. But $b(\bar{v}) < 0$ by the "no-easy-money condition" (4), since $\bar{v} \neq \bar{v}^0$ by assumption. Therefore, since we already have that $\overline{\lim}_{n\to\infty} b(\bar{v}^{(n)}) \leqslant b(\bar{v})$, 0 cannot be a limit point to the sequence $\langle b(\bar{v}^{(n)}) \rangle$, $\bar{v}^{(n)} \in D'$. Consequently, $b(\bar{v})$ for $\bar{v} \in D'$ is bounded away from zero, which implies that D must be bounded.

To prove convexity, let $\bar{v}''$ and $\bar{v}'''$ be two points in D. Then, for any $0 \leqslant \lambda \leqslant 1$,

$$\Pr\left\{\sum_{i=2}^{M} (\beta_i - r)\lambda v_i'' + \lambda r \geqslant 0\right\} = 1,$$

and

$$\Pr\left\{\sum_{i=2}^{M} (\beta_i - r)(1 - \lambda)v_i''' + (1 - \lambda)r \geqslant 0\right\} = 1,$$

which implies

$$\Pr\left\{\sum_{i=2}^{M} (\beta_i - r)(\lambda v_i'' + (1 - \lambda)v_i''') + r \geqslant 0\right\} = 1,$$

so that $\lambda\bar{v}'' + (1 - \lambda)\bar{v}''' \in D$. Thus, D is convex.

Let

$$\tilde{w}_n = \sum_{i=2}^{M} (\beta_i - r)v_i^n + r \qquad (n = 1, 2).$$

Then

$$h(\lambda\bar{v}^1 + (1 - \lambda)\bar{v}^2) = E[u(\lambda\tilde{w}_1 + (1 - \lambda)\tilde{w}_2)] \tag{32}$$

and

$$\lambda h(\bar{v}^1) + (1 - \lambda)h(\bar{v}^2) = \lambda E[u(\tilde{w}_1)] + (1 - \lambda)E[u(\tilde{w}_2)]. \tag{33}$$

For every pair of values $w_1 \neq w_2$ of the random variables $\tilde{w}_1$ and $\tilde{w}_2$ such that $\bar{v}^1$ and $\bar{v}^2 \in D$, we obtain, by the strict concavity of u,

$$u(\lambda w_1 + (1 - \lambda)w_2) > \lambda u(w_1) + (1 - \lambda)u(w_2), \quad 0 < \lambda < 1. \tag{34}$$

Consequently, (34) implies

$$E[u(\lambda\tilde{w}_1 + (1 - \lambda)\tilde{w}_2)] > \lambda E[u(\tilde{w}_1)] + (1 - \lambda)E[u(\tilde{w}_2)], \quad \bar{v}_1^1 \neq \bar{v}_2^2 \in D, \quad 0 < \lambda < 1,$$

which, by (32) and (33), in turn implies that h is strictly concave on D.

Since our problem has now been shown to be one of maximizing a strictly concave function over a nonempty, closed, bounded, convex set, it follows directly that the function h has a maximum and that the v_i^* are finite and unique.

A number of corollaries obtain from this lemma which we shall also require in the proof of Theorem 1.

COROLLARY 1: *Let $u(c)$, $\{\beta_i\}$, and r be defined as in the Lemma. Moreover, let $u(c)$ be such that it has no lower bound. Then the v_i^* which maximize* (31) *subject to* (27) *and* (28) *are such that*

$$Pr\left\{\sum_{i=2}^{M} (\beta_i - r)v_i^* + r > 0\right\} = 1.$$

The proof is immediate from the observation that $h \to -\infty$ as the greatest lower bound on b such that $\Pr \Sigma_{i=2}^{M} (\beta_i - r)v_i + r < b\} > 0$ approaches 0 from above.

COROLLARY 2: *Let $u(c)$, $\{\beta_i\}$, and r be defined as in the Lemma. Then the maximum of the function* (31) *subject to the constraints* (27) *and* (28) *is greater than or equal to $u(r)$.*

PROOF: When $v_i = 0$ for all i, which is always feasible, we obtain by (31) that $h = u(r)$.

COROLLARY 3: *Let $u(c)$, $\{\beta_i\}$, and r be defined as in the Lemma. Moreover, let $u(c)$ be such that $u(c) \leqslant b$. Then the vectors $\bar{v}$ which satisfy* (27) *and* (28) *are such that*

$$h(\bar{v}) \equiv E\left[u\left(\sum_{i=2}^{M} (\beta_i - r)v_i + r\right)\right] < b.$$

The proof is immediate from the observation that $u(c)$ is monotone increasing and that r, $\{\beta_i\}$, and the feasible v_i are bounded.

We are now ready to prove the theorem. The method of proof will be to verify that (22)–(25) is the (only) solution to (11).[8]

PROOF OF THEOREM 1 FOR MODELS I–II: Denote the right side of (11) by $T(x)$ upon inserting (22) for $f(x)$. This gives, for all decision points j,

$$T(x) = \max_{c,\{z_i\}} \left\{ \frac{1}{\gamma} c^\gamma + \alpha(1 - (\alpha k \gamma)^{1/(1-\gamma)})^{\gamma - 1} E\left[\frac{1}{\gamma} \left(\sum_{i=2}^{M} (\beta_i - r) z_i + r(x - c) + y + Y \right)^\gamma \right] \right\} \tag{35}$$

subject to

$$c \geqslant 0, \tag{12}$$

$$z_i \geqslant 0, \quad i \notin S, \tag{13}$$

and

$$\Pr\left\{ \sum_{i=2}^{M} (\beta_i - r) z_i + r(x - c) + y + (y/(r - 1)) \geqslant 0 \right\} = 1. \tag{14}$$

Since (14) may be written

$$\Pr\left\{ \sum_{i=2}^{M} (\beta_i - r) z_i + r(x + Y - c) \geqslant 0 \right\} = 1,$$

it follows from the "no-easy-money condition" (4) that (14) is satisfied if and only if *either*

$$s - c = 0 \tag{36}$$

and

$$z_i = 0 \qquad (i = 2, \ldots, M), \tag{37}$$

or

$$s - c > 0 \tag{38}$$

and

$$\Pr\left\{ \sum_{i=2}^{M} (\beta_i - r) z_i/(s - c) + r \geqslant 0 \right\} = 1, \tag{39}$$

where $s \equiv x + Y$.

Under feasibility with respect to (14), we then obtain

$$T(x) = \begin{cases} \max\left\{ \frac{1}{\gamma} s^\gamma, \overline{T}(x) \right\}, & 0 < \gamma < 1, \\ \max\left\{ -\infty, \overline{T}(x) \right\}, & \gamma < 0, \end{cases} \tag{40}$$

[8] A proof based on the method of successive approximations may be found in [7].

where

$$(41)\qquad \overline{T}(x) = \sup_{c,\{z_i\}}\left\{\frac{1}{\gamma}c^\gamma + \alpha(1 - (\alpha k\gamma)^{1/(1-\gamma)})^{\gamma-1}(s-c)^\gamma \times E\left[\frac{1}{\gamma}\left(\sum_{i=2}^{M}(\beta_i - r)z_i/(s-c) + r\right)^\gamma\right]\right\}$$

subject to (12), (38), (39), and

$$(42)\qquad z_i/(s-c) \geqslant 0, \quad i \notin S,$$

since (42) is equivalent to (13) in view of (38). But by (31) the expectation factor in (41) may be written

$$(43)\qquad h(z_2/(s-c), \ldots, z_M/(s-c))$$

and (26), the Lemma, and Corollary 2 give

$$(44)\qquad k\gamma \geqslant r^\gamma > 0 \quad \text{(Model I)},$$

$$(45)\qquad k\gamma \leqslant r^\gamma < 1 \quad \text{(Model II)},$$

while (26), the Lemma, and Corollary 3 give

$$(46)\qquad k\gamma > 0 \qquad \text{(Model II)}.$$

Thus, $\partial\overline{T}/\partial h > 0$ always in Model II and in Model I whenever

$$(47)\qquad k\gamma < \frac{1}{\alpha}$$

under feasibility. When $k\gamma > 1/\alpha$ in Model I, $\overline{T}(x)$ does not exist; when $k\gamma = 1/\alpha$, (41) and (40) give $T(x) = (1/\gamma)s^\gamma \neq f(x)$. Consequently, it remains to consider the case when $\partial\overline{T}/\partial h > 0$.

Since the maximum of (43) subject to (42) and (39) is k by (26) and the Lemma, we obtain by the Lemma that the strategy

$$\frac{z_i}{s-c} = v_i^* \qquad (i = 2, \ldots, M)$$

or

$$(48)\qquad z_i^* = v_i^*(s-c) \qquad (i = 2, \ldots, M)$$

is optimal and unique for *every* c which satisfies (12) and (38) when (38) holds. It is clearly also optimal when (36) and (37) hold. Consequently, (40) reduces to

$$(49)\qquad T(x) = \max_{0 \leqslant c \leqslant s}\left\{\frac{1}{\gamma}c^\gamma + \alpha k(1 - (\alpha k\gamma)^{1/(1-\gamma)})^{\gamma-1}(s-c)^\gamma\right\}.$$

Since $u(c)$ is strictly concave and $u'(0) = \infty$ in Models I and II, $T(x)$ is strictly concave and differentiable with an "interior" unique solution $c^*(x)$ whenever

$$(50)\qquad \alpha k(1 - (\alpha k\gamma)^{1/(1-\gamma)})^{\gamma-1}\begin{cases} > 0 & \text{(Model I)}, \\ < 0 & \text{(Model II)}, \end{cases}$$

and $s \geqslant 0$. In this case, setting $dT/dc = 0$ and solving for c, we get,

$$c^{\gamma-1} - \alpha k\gamma(1 - (\alpha k\gamma)^{1/(1-\gamma)})^{\gamma-1}(s - c)^{\gamma-1} = 0$$

or

$$c^*(x) = (1 - (\alpha k\gamma)^{1/(1-\gamma)})(x + Y). \tag{51}$$

In Model I, (50) is satisfied whenever (44) and (47) hold; as noted earlier, no solution exists in Model I for those cases in which $k\gamma \geqslant 1/\alpha$. In Model II, (50) is always satisfied as seen from (45) and (46).

Inserting (51) in (48) we obtain

$$z_i^*(x) = (\alpha k\gamma)^{1/(1-\gamma)} v_i^*(x + Y) \qquad (i = 2, \ldots, M)$$

and (24) follows from (6) upon insertion of $c^*(x)$ and the $z_i^*(x)$. $T(x)$ now becomes, upon insertion of $c^*(x)$ in (49),

$$\begin{aligned} T(x) &= \frac{1}{\gamma}(1 - (\alpha k\gamma)^{1/(1-\gamma)})^{\gamma} s^{\gamma} + \alpha k(1 - (\alpha k\gamma)^{1/(1-\gamma)})^{\gamma-1} s^{\gamma}(\alpha k\gamma)^{\gamma/(1-\gamma)} \\ &= \frac{1}{\gamma}(1 - (\alpha k\gamma)^{1/(1-\gamma)})^{\gamma-1} s^{\gamma} \\ &= f(x) \end{aligned}$$

and the solution clearly exists for $s \geqslant 0$ or

$$x_j \geqslant -Y. \tag{52}$$

Since (52) is an induced constraint with respect to period $j - 1$, it remains to be verified that (52) is either redundant or not effective in period $j - 1$. Because (52) is already present in period $j - 1$ through (14), the induced constraint (52) is redundant, which completes the proof.

Proof of Theorem 1 for Model III: Denote the right side of (11) $T(x)$ upon inserting (22) for $f(x)$. This gives, for all decision points j,

$$\begin{aligned} T(x) = \max_{c,\{z_i\}} \Bigg\{ \log c + \frac{\alpha}{1-\alpha} E\Bigg[\log\Bigg(\sum_{i=2}^{M} (\beta_i - r)z_i \\ + r(x - c) + y + Y\Bigg)\Bigg] + K\Bigg\} \end{aligned}$$

where

$$K \equiv \frac{\alpha}{1-\alpha}\log(1-\alpha) + \frac{\alpha^2 \log\alpha}{(1-\alpha)^2} + \frac{\alpha^2 k}{(1-\alpha)^2}$$

subject to (12), (13), and (14). By the reasoning for Models I and II, we obtain

$$T(x) = \max\{-\infty, \overline{T}(x)\} \tag{53}$$

where

(54) $$\overline{T}(x) = \sup_{c,\{z_i\}} \left\{ \log c + \frac{\alpha}{1-\alpha} \log (s - c) + \frac{\alpha}{1-\alpha} E\left[\log \left(\sum_{i=2}^{M} (\beta_i - r) z_i/(s - c) + r \right) \right] \right\} + K$$

subject to (12),

(38) $$s - c > 0,$$

(42) $$z_i/(s - c) \geqslant 0 \qquad (i = 2, \ldots, M),$$

and

(39) $$\Pr \left\{ \sum_{i=2}^{M} (\beta_i - r) z_i/(s - c) + r \geqslant 0 \right\} = 1 .$$

By (31), the next to last term in (54) can be written

(55) $$\frac{\alpha}{1-\alpha} h(z_2/(s - c), \ldots, z_M/(s - c))$$

where $\partial \overline{T}/\partial h > 0$. Since the maximum of (55) subject to (42) and (39) is $(\alpha k/1 - \alpha)$ by (26) and the Lemma, we obtain from the Lemma that

(48) $$z_i^*(x) = v_i^*(x + Y - c) \qquad (i = 2, \ldots, M)$$

is optimal and unique for *every* c which satisfies (12) and (38). Thus, (53) reduces, in analogy with Models I and II, to

(56) $$T(x) = \max_{0 \leqslant c \leqslant s} \left\{ \log c + \frac{\alpha}{1-\alpha} \log (s - c) + \frac{\alpha k}{1-\alpha} + K \right\}$$

where $T(x)$ always exists since $0 < \alpha < 1$; furthermore, $T(x)$ is strictly concave and differentiable. Setting $\partial T/\partial c = 0$ we obtain

(57) $$c^*(x) = (1 - \alpha)(x + Y),$$
$$z_i^*(x) = \alpha v_i^*(x + Y) \qquad (i = 2, \ldots, M),$$

and (24), all unique. Inserting (57) into (56) gives

$$T(x) = \log (1 - \alpha) + \log s + \frac{\alpha}{1-\alpha} \log \alpha + \frac{\alpha}{1-\alpha} \log s + \frac{\alpha k}{1-\alpha} + \frac{\alpha}{1-\alpha} \log (1 - \alpha) + \frac{\alpha^2 \log \alpha}{(1-\alpha)^2} + \frac{\alpha^2 k}{(1-\alpha)^2}$$
$$= f(x).$$

Since $f(x)$ exists for $x_j \geqslant -Y$, which as an induced constraint with respect to period $j - 1$ is made redundant by (14) for that period, the proof is complete.

When $y = 0$, the solution to (11) reduces to

$$f(x) = Au(x) + C,$$
$$c^*(x) = Bx,$$
$$z_1^*(x) = (1 - B)(1 - v^*)x,$$
$$z_i^*(x) = (1 - B)v_i^*x \qquad (i = 2, \ldots, M).$$

But then, letting $s \equiv x + Y$,

$$f(s) = Au(s) + C,$$
$$c^*(s) = Bs,$$
$$z_1^*(s) = (1 - B)(1 - v^*)s,$$
$$z_i^*(s) = (1 - B)v_i^*s \qquad (i = 2, \ldots, M).$$

As a result, except for $z_1^*(x + Y)$, the solution to the original problem is not altered when the individual, instead of receiving the noncapital income stream in installments, is given its present value Y in advance. Thus, instead of letting x be the state variable when there is a noncapital income, one could let $x + Y$ be the state variable (pretending there is no income), as long as Y is deducted from $z_1^*(x + Y)$.

Note that it is sufficient, though not necessary, for a solution *not* to exist in Model I that $r^\gamma \geqslant 1/\alpha$ (Corollary 2).

THEOREM 2: *Let α, $\{\beta_i\}$, r, y, and Y be defined as in Section 2. Moreover, let $u(c) = -e^{-\gamma c}$ for $c \geqslant 0$ where $\gamma > 0$. Then a solution to* (11) *subject to* (12)–(14) *exists for $x \geqslant -Y + [r/(\gamma(r-1)^2)] \log(-\alpha k r)$ and is given by*

$$f(x) = -\frac{r}{r-1}(-\alpha k r)^{1/(r-1)} e^{-[\gamma(r-1)/r](x+Y)}, \tag{58}$$

$$c^*(x) = \frac{r-1}{r}(x + Y) - \frac{1}{\gamma(r-1)} \log(-\alpha k r), \tag{59}$$

$$z_1^*(x) = \frac{x}{r} - \frac{y}{r} + \frac{\log(-\alpha k r) - r v^*}{\gamma(r-1)}, \tag{60}$$

$$z_i^*(x) = \frac{r}{\gamma(r-1)} v_i^* \qquad (i = 2, \ldots, M), \tag{61}$$

where the constants k and v_i^* ($v^* \equiv \sum_{i=2}^M v_i^*$) are given by

$$k \equiv E[-e^{-\sum_{i=2}^M(\beta_i - r)v_i^*}] = \max_{\{v_i\}} E[-e^{-\sum_{i=2}^M(\beta_i - r)v_i}] \quad \textit{subject to } (27) \tag{62}$$

provided that

$$\log(-\alpha k r) + b(\bar{v}^*) \geqslant 0 \tag{63}$$

where $b(\bar{v}^*)$ is the greatest lower bound on b such that

$$Pr\left\{\sum_{i=2}^{M}(\beta_i - r)v_i^* < b\right\} > 0$$

and $\bar{v}^* \equiv (v_2^*, \ldots, v_M^*)$. Moreover, the solution is unique.

Since the conditions under which Theorem 2 holds are quite restrictive, the reader is referred to [7] for the proof. Condition (63) insures that the individual's capital position x is nondecreasing over time with probability 1; it must hold for a solution to exist in closed form. The condition $\alpha r \geqslant 1$ is a necessary, but not sufficient, condition for (63) to be satisfied.[9]

4. PROPERTIES OF THE OPTIMAL CONSUMPTION STRATEGIES

In each of the four models we note that the optimal consumption function $c^*(x)$ is linear increasing in capital x and in noncapital income y. Whenever $y > 0$, positive consumption is called for even when the individual's net worth is negative, as long as it is greater than $-Y$ in Models I–III and greater than $-Y + [r/(\gamma(r-1)^2)] \log(-\alpha kr)$ in Model IV. Only *at* these end points would the individual consume nothing.

Since $x + Y$ may be viewed as permanent (normal) income and consumption is proportional $(0 < B < 1)$ to $x + Y$ in Models I–III, we see that the optimal consumption functions in these models satisfy the permanent (normal) income hypotheses precisely [9, 5, 3].

In each model, $c^*(x)$ is decreasing in α. Thus, the greater the individual's impatience $1 - \alpha$ is, the greater his present consumption would be. This, of course, is what we would expect.

By (20) and (21), the relative and absolute risk aversion indices of Models I–IV are as follows:

$$q^*(c) = 1 - \gamma \quad \text{(Models I–II)},$$
$$q^*(c) = 1 \quad \text{(Model III)},$$
$$q(c) = \gamma \quad \text{(Model IV)}.$$

[9] For example, when $u(c) = -e^{-.0001c}$, $\alpha = .99$, $y = \$10{,}000$, $r = 1.06$, $M = 2$, and β_2 assumes each of the values .96 and 1.17 with probability .5, a solution exists for $x \geqslant \$-22{,}986$. For selected capital positions, the optimal amounts to consume, lend, and invest in this case are as follows:

x	$c^*(x)$	$z_1^*(x)$	$z_2^*(x)$
\$ −22,986	0	\$− 102,488	\$79,502
0	\$ 1,301	− 80,803	79,502
50,000	4,131	− 33,633	79,502
100,000	6,961	13,537	79,502
500,000	29,601	390,897	79,502
1,000,000	57,901	862,597	79,502

The maximum loss in each period from risky investment is \$3,180.

In Models I–II, we obtain

$$(64)\qquad \frac{\partial B}{\partial(1-\gamma)} = -(\alpha k\gamma)^{1/(1-\gamma)}\left\{\frac{[d(k\gamma)/d(1-\gamma)]}{k\gamma(1-\gamma)} - \frac{\log(\alpha k\gamma)}{(1-\gamma)^2}\right\}$$

where $d(k\gamma)/d(1-\gamma)$ is negative whenever $b(\bar{v}^*) \geqslant 1 - r$; otherwise the sign is ambiguous. Since $k\gamma > 0$ and $\alpha k\gamma < 1$, the sign of (64) is ambiguous in both cases; i.e., a change in relative risk aversion may either decrease or increase present consumption. In Model IV, on the other hand, $c^*(x)$ is increasing in γ; i.e., a more risk averse individual consumes more, ceterus paribus.

From (26) and (62) we observe that k is a natural measure of the "favorableness" of the investment opportunities. This is because k is a maximum determined by (the one-period utility function and) the distribution function (F); moreover, F is reflected in the solution only through k, and $f(x)$ is increasing in k. Let us examine the effect of k on the marginal propensities to consume out of capital and non-capital income.

Equation (29) gives

$$\frac{\partial B}{\partial k} = \frac{\alpha\gamma}{\gamma - 1}(\alpha k\gamma)^{\gamma/(1-\gamma)}\begin{cases} < 0 & \text{(Model I)},\\ > 0 & \text{(Model II)}.\end{cases}$$

Thus, we find that the propensity to consume is *decreasing* in k in the case of Model I. This phenomenon can at least in part be attributed to the fact that the utility function is bounded from below but not from above; the loss from postponement of current consumption is small compared to the gain from the much higher rate of consumption thereby made possible later. In Model II, on the other hand, where the utility function has an upper bound but no lower bound, the optimal amount of present consumption is *increasing* in k, which seems more plausible from an intuitive standpoint.

In Model III, we observe from (30) the curious phenomenon that the optimal consumption strategy is independent of the investment opportunities in every respect. While the marginal propensity to consume is independent of k in Model IV also, the *level* of consumption in this case is an increasing function of k as is apparent from (59). We recall that the utility function in Model III is unbounded while that in Model IV is bounded both from below and from above. Thus, the class of utility functions we have examined implies an exceptionally rich pattern of consumption behavior with respect to the "favorableness" of the investment opportunities.

5. PROPERTIES OF THE OPTIMAL INVESTMENT AND BORROWING-LENDING STRATEGIES

The properties exhibited by the optimal investment strategies are in a sense the most interesting. Turning first to Model IV, we note that the portfolio of productive investments is constant, both in mix and amount, at all levels of wealth. The optimal portfolio is also independent of the noncapital income stream and the level of impatience $1 - \alpha$ possessed by the individual, as shown by (61) and (62).

Similarly, we find in Models I–III that, since for all $i, m > 1$, $z_i^*(x)/z_m^*(x) = v_i^*/v_m^*$ (which is a constant), the *mix* of risky investments is independent of wealth, noncapital income, and impatience to spend. In addition, the *size* of the total investment commitment in each period is proportional to $x + Y$. We also note that when $y = 0$, the ratio that the risky portfolio $\Sigma_{i=2}^{M} z_i^*(x)$ bears to the total portfolio $\Sigma_{i=1}^{M} z_i^*(x)$ is independent of wealth in each model.

In summary, then, we have the surprising result that the optimal mix of risky (productive) investments in each of Models I–IV is independent of the individual's wealth, noncapital income stream, and rate of impatience to consume; the optimal mix depends in each case only on the probability distributions of the returns, the interest rate, and the individual's one-period utility function of *consumption*.

In each case, we find that lending is linear in wealth. Turning first to Models I–III, we find that borrowing always takes place at the lower end of the wealth scale; (24) evaluated at $x = -Y$ gives $-Y < 0$ as the optimal amount to lend. From (24) we also find that $z_1^*(x)$ is increasing in x if and only if $1 - v^* > 0$ since $1 - B$ is always positive. As a result, the models always call for borrowing at least when the individual is poor; whenever $1 - v^* > 0$, they also always call for lending when he is sufficiently rich.

In Model IV, we observe that lending is always increasing in x. Thus, when an individual in this model becomes sufficiently wealthy, he will always become a lender. At the other extreme, when x is at the lower boundary point of the solution set, he will generally be a borrower, though not necessarily, since $z_1^*(x)$ evaluated at $x = -Y + [r/(\gamma(r-1)^2)] \log(-\alpha k r)$ gives

$$-Y + \frac{r \log(-\alpha k r)}{\gamma(r-1)^2} - \frac{r v^*}{\gamma(r-1)}$$

which may be either negative or positive.

We shall now consider the case when the lending rate differs from the borrowing rate as is usually the case in the real world. Let $r_B - 1$ and $r_L - 1$ denote the borrowing and lending rates, respectively, where $r_B > r_L$. Unfortunately, the sign of dv^*/dr is not readily determinable. However, since $f(x)$ is increasing in k, the analysis is straight-forward.[10]

[10] When $r_B > r_L$, the "no-easy-money condition" requires that the joint distribution function of $\beta_2, \ldots, \beta_M$ satisfies

$$\text{(4a)} \qquad \Pr\left\{\sum_{i=2}^{M} (\beta_i - r_B)\theta_i < 0\right\} > 0$$

for all finite numbers $\theta_i \geqslant 0$ such that $\theta_i > 0$ for at least one i;

$$\text{(4b)} \qquad \Pr\left\{\sum_{i \notin S} (\beta_i - r_L)\theta_i < 0\right\} > 0$$

for all finite numbers $\theta_i \leqslant 0$ such that $\theta_i < 0$ for at least one i; and

$$\text{(4c)} \qquad \Pr\left\{\sum_{\substack{i=2\\ i \notin S^*}}^{M} \beta_i\theta_i - \sum_{k \in S^*} \beta_k\theta_k < 0\right\} > 0$$

for all finite numbers $\theta_i, \theta_k \geqslant 0$ and all $S^* \subseteq S$ such that

$$\sum_{\substack{i=2\\ i \notin S^*}}^{M} \theta_i = \sum_{k \in S^*} \theta_k,$$

and $\theta_i > 0$ for at least one i. When $r_B = r_L$, 4(a)–4(c) reduce to (4).

Consider first Models I–III when noncapital income $y = 0$. In that case, it is apparent from (24) that when the individual is not in the trapping state (i.e., $x > -Y$), he either always borrows, always lends, or does neither, depending on whether $1 - v^*$ is negative, positive, or zero. Let k_L denote the maximum of (31) when the lending rate is used and the constraint

$$\sum_{i=2}^{M} v_i \leqslant 1 \tag{65}$$

is added to constraints (27) and (28). Since the set of vectors $\bar{v}$ which satisfy (65) is convex and includes $\bar{v} = (0, \ldots, 0)$, the Lemma still holds when (65) is added to the constraint set. Analogously, let k_B denote the maximum of (31) under the borrowing rate r_B subject to (27), (28), and

$$\sum_{i=2}^{M} v_i \geqslant 1. \tag{66}$$

Again, the Lemma holds since the set of $\bar{v}$ satisfying (66) is convex and any $\bar{v}$ such that $\Sigma_{i=2}^{M} v_i = 1$, $v_i \geqslant 0$, for example, satisfies all constraints. Setting $k \equiv \max\{k_B, k_L\}$, Theorem 1 holds as before when $y = 0$.

When $y > 0$ in Models I–III and in the case of Model IV, no "simple" solution appears to exist when $r_B > r_L$.

6. THE BEHAVIOR OF CAPITAL

We shall now examine the behavior of capital implied by the optimal investment and consumption strategies of the different models. According to one school, capital growth is said to exist whenever

$$E[x_{j+1}] > x_j \qquad (j = 1, 2, \ldots), \tag{67}$$

that is, capital growth is defined as expected growth [**10**]. We shall reject this measure since under this definition, as $j \to \infty$, x_j may approach a value less than x_1 with a probability which tends to 1. We shall instead define growth as asymptotic growth; that is, capital growth is said to exist if

$$\lim_{j \to \infty} \Pr\{x_j > x_1\} = 1. \tag{68}$$

When the $>$ sign is replaced by the $\geqslant$ sign, we shall say that we have capital nondecline. If there is statistical independence with respect to j, (67) is implied by (68) but the converse does not hold, as noted.

Model IV will be considered first. From (63) it follows that nondecline of capital is always implied (in fact, the solution to the problem is contingent upon the condition that capital does not decrease, as pointed out earlier). It is readily seen that a sufficient, but not necessary, condition for growth is that there be a nonzero investment in at least one of the risky investment opportunities since in that case $\Pr\{x_{j+1} > x_j\} > 0, j = 1, 2, \ldots$, by (63). A necessary and sufficient condition for asymptotic capital growth is $\alpha r > 1$, which is readily verified by reference to (62), (63), and the foregoing statement.

Let us now turn to Models I–III and let, as before, $s_j \equiv x_j + Y$. From (7), (23), and (25) we now obtain

$$(69) \qquad \begin{aligned} s_{j+1} &= s_j(1 - B)\left[\sum_{i=2}^{M} (\beta_i - r)v_i^* + r\right] \\ &= s_j W \qquad\qquad (j = 1, 2, \ldots) \end{aligned}$$

where W is a random variable. By (28), $W \geqslant 0$. Attaching the subscript n to W for the purpose of period identification, we note that since

$$(70) \qquad s_j = s_1 \prod_{n=1}^{j-1} W_n,$$

(70) verifies that

$$s_j \geqslant 0 \quad \text{for all } j \text{ whenever } s_1 \geqslant 0 \quad \text{(Models I–III)}.$$

Moreover, since $\Pr\{W > 0\} = 1$ in Models II and III by Corollary 1, it follows that

$$(71) \qquad s_j > 0 \quad \text{whenever } s_1 > 0 \text{ for all finite } j \quad \text{(Models II–III)}.$$

From (70) we also observe that $s_j = 0$ whenever $s_k = 0$ for all $j > k$. Consequently, $x = -Y$ is a trapping state which, once entered, cannot be left. In this state, the optimal strategies in each case call for zero consumption, no productive investments, the borrowing of Y, and the payment of noncapital income y as interest on the debt. In Models II and III, it follows from (71) that the trapping state will never be reached in a finite number of time periods if initial capital is greater than $-Y$.

Equation (70) may be written

$$s_j = s_1 e^{\sum_{n=1}^{j-1} \log W_n}.$$

The random variable $\sum_{n=1}^{j-1} \log W_n$ is by the Central Limit Theorem asymptotically normally distributed; its mean is $(j-1)E[\log W]$. By the law of large numbers,

$$\frac{\sum_{n=1}^{j-1} \log W_n}{j - 1} \to E[\log W] \quad \text{as } j \to \infty.$$

Thus, since $s_j > s_1$ if and only if $x_j > x_1$, it is necessary and sufficient for capital growth to exist that $E[\log W] > 0$.

It is clear that μ given by $\mu \equiv e^{E[\log W]}$ may be interpreted as the mean growth rate of capital. By (69), we obtain

$$E[\log W] = \log(1 - B) + E\left[\log\left\{\sum_{i=2}^{M} (\beta_i - r)v_i^* + r\right\}\right].$$

For Model III, this becomes, by (30) and (26),

$$E[\log W] = \log \alpha + \max_{\{v_i\}} E\left[\log \left\{ \sum_{i=2}^{M} (\beta_i - r)v_i + r \right\} \right]$$

subject to (27) and (28). Thus, a person whose one-period utility function of consumption is logarithmic will always invest the capital available after the allotment to current consumption so as to maximize the mean growth rate of capital plus the present value of the noncapital income stream.

7. GENERALIZATIONS

We shall now generalize the preceding model to the nonstationary case. We then obtain, by the same approach as in the stationary case, for all j,

$$f_j(x_j) = \max_{c_j, \{z_{ij}\}} \left\{ u(c_j) + \alpha_j E\left[f_{j+1}\left(\sum_{i=2}^{M_j} (\beta_{ij} - r_j)z_{ij} + r_j(x_j - c_j) + y_j \right) \right] \right\} \tag{72}$$

subject to

$$c_j \geqslant 0, \tag{73}$$

$$z_{ij} \geqslant 0, \quad i \notin S_j, \tag{74}$$

and

$$\Pr\{x_{j+1} \geqslant -Y_{j+1}\}, \tag{75}$$

where the patience factor α, the number of available investment opportunities M and S and their random returns $\beta_i - 1$, the interest rate r, and the noncapital income y may vary from period to period; this, of course, requires that they be time identified through subscript j. Time dependence on the part of any *one* of the preceding parameters also requires that $f(x)$ be subscripted.

As shown in [7], the solution to the nonstationary model is qualitatively the same as the solution to the stationary model.

In the case of a finite horizon, the problem again reduces to (72)–(75) with $f_{n+1}(x_{n+1}) \equiv 0$ if the horizon is at decision point $n+1$. In this case, $f(x)$, x, c, z_i, and Y must clearly be time identified through subscript j even in the stationary model. Under a finite horizon, a solution always exists even for Model I. Again, the solution is qualitatively the same as in the infinite horizon case except that the constant of consumption proportionality B_j increases with time j, $B_n = 1$, and $z_{in}^* = 0$ for all i.[11]

University of California, Berkeley

Manuscript received September, 1966; *revision received January*, 1969.

[11] The implications of the results of the current paper with respect to the theory of the firm may be found in [6].

REFERENCES

[1] ACZÉL, J.: *Lectures on Functional Equations and Their Applications*. New York, Academic Press, 1966.

[2] BELLMAN, RICHARD: *Dynamic Programming*. Princeton, Princeton University Press, 1957.

[3] FARRELL, M. J.: "The New Theories of the Consumption Function," *Economic Journal*, December, 1959.

[4] FISHER, IRVING: *The Theory of Interest*. New York, MacMillan, 1930; reprinted, Augustus Kelley, 1965.

[5] FRIEDMAN, MILTON: *A Theory of the Consumption Function*. Princeton, Princeton University Press, 1957.

[6] HAKANSSON, NILS: "An Induced Theory of the Firm Under Risk: The Pure Mutual Fund," *Journal of Financial and Quantitative Analysis*, June 1970.

[7] ———: "Optimal Investment and Consumption Strategies for a Class of Utility Functions," Ph.D. Dissertation, University of California at Los Angeles, 1966; also, Working Paper No. 101, Western Management Science Institute, University of California at Los Angeles, June, 1966.

[8] ———: "Risk Disposition and the Separation Property in Portfolio Selection," *Journal of Financial and Quantitative Analysis*, December, 1969.

[9] MODIGLIANI, F., AND R. BRUMBERG: "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," *Post-Keynesian Economics* (ed. K. Kurihara), New Brunswick, Rutgers University Press, 1954.

[10] PHELPS, EDMUND: "The Accumulation of Risky Capital: A Sequential Utility Analysis," *Econometrica*, October, 1962.

[11] PRATT, JOHN: "Risk-Aversion in the Small and in the Large," *Econometrica*, January–April, 1964.

[12] VON NEUMANN, JOHN, and OSKAR MORGENSTERN: *Theory of Games and Economic Behavior*. Princeton University Press, 1947.