

OPTIMAL ENTREPRENEURIAL DECISIONS IN A COMPLETELY STOCHASTIC ENVIRONMENT*

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This paper develops a normative model of the entrepreneur's decision problem in which the following elements are stochastic: the entrepreneur's preferences, his lifetime, the returns from investments, and the process obeyed by the interest rate. Furthermore, the entrepreneur's preferences are assumed to be sensitive to the opportunities facing him at each decision point as well as other environmental factors.

At each decision point the entrepreneur must decide how to allocate his resources between consumption, life insurance, various investment opportunities, and lending/borrowing. His objective is postulated to be the maximization of expected utility from consumption as long as he lives and from the bequest left upon his death.

Optimal decision functions are obtained in closed form for a class of utility functions; their properties are examined and compared to those of the optimal strategies of less general models.

1. Introduction and Summary

In two previous papers [8] and [9], a family of normative models of the individual's economic decision problem under risk was presented. At the same time, certain implications of these models with respect to individual behavior were deduced for a class of utility functions. In this paper, these models are generalized in the sense that both preferences and opportunities are permitted to be stochastic as well as interdependent; furthermore, preferences are assumed to be sensitive to environmental factors generally.

The first paper to consider the basic decision problem treated in this article is due to Phelps, who examined the case when there is one type of investment yielding a stochastic return [20]. Somewhat later, Yaari analyzed the case when the individual's lifetime is uncertain [25] but investment yields are deterministic. Models incorporating choice among several risky assets as well as a riskless one were then developed by Hakansson [7] (this model is summarized in [8]), Leland [14], Levhari and Srinivasan [15], Samuelson [22], and Merton [18]. Hakansson also studied the case when both investment returns and the horizon are uncertain [9]. In the present paper, all of the following elements are stochastic: the individual's preferences, his lifetime, the returns from investments, and the process obeyed by the interest rate. The model developed may be viewed as an application of the state-preference approach [11] to the sequential case.

In §2, the various components of the basic decision model used in the present paper are constructed. The individual's objective is postulated to be the joint maximization of expected utility from consumption as long as he lives and from the bequest left upon his death. His lifetime is presumed to be a random variable and his preferences are permitted to depend on future states of the economy. This dependence arises via the individual's impatience rates, which are subject to influence by the environment. The economy may obey a finite-state, discrete-time stochastic process of any generality

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and each state sequence may offer different opportunities. Since the individual's resources are assumed to consist of only an initial capital position, the individual in this paper may be thought of as a self-employed businessman.

In addition to insurance available at a "fair" rate, the entrepreneur faces both financial opportunities (borrowing and lending) and an arbitrary number of productive investment opportunities. The interest rate is presumed to be positive but may depend on the state of the world and the borrowing rate may exceed the lending rate. The returns from the productive opportunities are assumed to be random variables, whose probability distributions may depend on both time and the state transition but are assumed to satisfy the "no-easy-money-condition." While no limit is placed on borrowing, the individual is required to be solvent at the time of his death with probability 1; that is, all debt must be fully secured at all times.

The components developed in §2 are assembled into a formal model in §3. The fundamental characteristic of the approach taken is that the portfolio composition decision, the financing decision, the insurance decision, and the consumption decision are all analyzed in *one* model. The model may therefore be viewed as a formalization of Irving Fisher's model of the individual under risk, as given in *The Theory of Interest* [4]. The vehicle of analysis is discrete-time dynamic programming.

§4 contains some preliminary results which are used in §5 to find the solution to the entrepreneur's decision problem for that class of one-period utility functions which exhibits constant relative risk aversion.

The optimal consumption strategies are examined in §6 in relation to the permanent income hypothesis and with respect to their sensitivity to changes in impatience, age, risk aversion, favorableness of future opportunities, and their reflection of attitudes toward "the Joneses." The properties of the optimal borrowing/lending strategies are considered in §7. In several of these investigations, the optimal strategies are found to exhibit rather diverse patterns within the class of utility functions examined.

In §8, the optimal mix of risky assets is found to be independent of wealth in each period. However, the optimal mix is not independent of age, the survival probabilities, the impatience to consume, the strength of the bequest motive, and for a subset of models, the future investment opportunities, as in the case when preferences are independent of opportunities and investment returns are stochastically independent over time [6], [9]. It is also noted that the model considered can cope with long-term investments whose returns obey the most general discrete-time stochastic process.

In §9, it is found that the entrepreneur should purchase life insurance at a "fair" rate only when the time structure of his bequest motive is highly irregular. At the same time, he can in most instances increase his expected utility by selling insurance on the lives of others. Finally, §10 considers the model in the context of multiperson firms and relates additive but opportunity-dependent utility functions to nonadditive opportunity-independent utility functions.

2. Assumptions and Notation

In this section, the postulates concerning the individual's preferences, resources, and opportunities will be specified. As the various building blocks are introduced, we also give the notation to be used in the following sections.

2.1 Resources and Opportunities

We assume that the entrepreneur has the opportunity to make decisions at discrete points, called decision points, which are equally spaced in time. The space of

time intervening between the two adjacent decision points j and $j + 1$ will be denoted period j .

Let $\bar{p}_j > 0$ be the individual's probability of passing away in the j th period, $j = 1, \dots, J$, where $\sum_{j=1}^J \bar{p}_j = 1$; thus J is the last period in which death may occur. We now observe that

$$(1) \quad p_{mj} \equiv \bar{p}_j / \sum_{k=m}^J \bar{p}_k \quad m, j = 1, \dots, J, \quad m \leq j$$

expresses the probability that the entrepreneur will pass away in period j given that he is alive at the beginning of period m .

We denote the amount of the individual's monetary (capital) resources at the j th decision point, given that he is alive at that point, by x_j . For simplicity, we shall view all of his capital as belonging to his business. In the event the individual passes away in period $j - 1$, the amount of his resources at the end of that period will be termed his estate and will be designated x'_j .

We assume that the individual may also be the recipient of a noncapital income stream during all or part of his lifetime. However, since we consider the individual to be a self-employed businessman, we will assume that the only possible source of this income is the labor he devotes to his own business; his salary is therefore paid from his own capital. Thus, the amount he pays himself in salary plus net nonsalary withdrawals, which we shall call consumption, is a nontrivial decision variable in the sense that it represents net permanent capital outflow from his business. As a result, his choice of salary is completely arbitrary except possibly under certain tax structures (which we shall not consider) and need not be considered separately from other withdrawals.

We postulate that the individual faces both financial and productive opportunities in each period. The first of these is the opportunity to borrow or lend arbitrary amounts of money in each period at a riskless rate on the condition that all debt (including interest) is fully secured. The amount saved at decision point j will be denoted z_{1j} ; negative z_{1j} will then indicate borrowing. The productive opportunities faced by the individual consist of the possibility of making risky investments.

It will be assumed that the returns from the productive opportunities in a given period depend, among other things, on the change in the "general condition" of the economic environment (which we shall call the economy) that takes place in that period. More specifically, we assume that N_j different states of the economy can be perceived with respect to decision point j and that the probability of a transition from state m to state n in period j , which we denote p_{jmn} (≥ 0 , $\sum_{n=1}^{N_j+1} p_{jmn} = 1$), is known for all j, m , and n . We further assume that the transition probabilities are constants, i.e. independent of past transitions, which is equivalent to saying that the economy obeys a (possibly) nonstationary Markov process.

We denote the interest rate in period j , given that the economy at the beginning of the period is in state m , by $r_{jm} - 1$, which is assumed to be a positive constant. This is equivalent to assuming that enough states can be distinguished in advance by the individual for the interest rate in each period, conditioned on the state of the economy at the beginning of the period, to have a degenerate distribution.

Let the total number of different risky (productive) opportunities available to the individual at decision point j when the economy is in state m be $M_{jm} - 1$, of which the first $S_{jm} - 1 \leq M_{jm} - 1$ may be sold short. A short sale will be defined as the opposite of a long investment; that is, if the individual sells opportunity i short in the amount θ , he will receive θ immediately (to do with as he pleases) in return for the obligation to pay the transformed value of θ at the end of the period. The net proceeds realized

when the economy is in state n at the end of period j from each unit of capital invested in opportunity i when the economy was in state m at the beginning of that period will be denoted β_{ijmn} , where β_{ijmn} is a random variable. Thus, returns to scale are assumed to be stochastically constant, all investments are assumed to be realizable in cash at the end of each period, and taxes and conversion costs, if any, are assumed to be proportional to the amount invested. Furthermore, the returns from risky assets in a given period are allowed to depend on the state of the economy both at the beginning *and* at the end of the period. The reason for making this allowance, of course, is that a transition from “recovery” to “boom” need not offer the same opportunities at a given moment in time as a transition from “boom” to “boom”, for example. The amount invested in opportunity i , $i = 2, \dots, M_{jm}$, at the j th decision point will be denoted z_{ij} and is, as indicated earlier, a decision variable along with z_{1j} .

It will be assumed that the joint distribution functions F_{jmn} given by

$$(2) \quad F_{jmn}(x_2, x_3, \dots, x_{M_{jm}}) \equiv \Pr \{ \beta_{2jmn} \leq x_2, \beta_{3jmn} \leq x_3, \dots, \beta_{M_{jm}jn} \leq x_{M_{jm}} \}$$

all j, m , and n

are known and independent.^{1,2} In addition, we shall postulate that the $\{\beta_{ijmn}\}$ satisfy the following conditions:

$$(3) \quad \Pr \{ 0 \leq \beta_{ijmn} < \beta_{jm} \} = 1, \quad i = 2, \dots, M_{jm}, \quad \text{for } j, m, \text{ and } n, \beta_{jm} > r_{jm}$$

$$(4) \quad \Pr \{ \sum_{i=2}^{M_{jm}} (\beta_{ijmn} - r_{jm})\theta_i < 0 \} > 0$$

for all j , all m , some n for which $p_{jmn} > 0$, and all finite θ_i such that $\theta_i \geq 0$ for all $i > S_{jm}$ and $\theta_i \neq 0$ for at least one i . (4) is a modification of the “no-easy-money-condition” for the case when the lending rate equals the interest rate [7]. This condition states that no combination of productive investment opportunities exists in any period which provides, with probability 1, a return at least as high as the (borrowing) rate of interest; no combination of short sales is available in which the probability is zero that a loss will exceed the (lending) rate of interest; and no combination of productive investments made from the proceeds of any combination of short sales can guarantee against loss. (4) may be viewed as a condition which the prices of all assets must satisfy in equilibrium.

We shall also provide the individual with the opportunity to purchase term insurance on his own life and to sell (purchase) term insurance on the lives of others in each period. Let $t_j \geq 0$ denote the premium paid by the individual at the j th decision point for life insurance *on his own life* during period j . If the individual passes away during this period, which by (1) has probability p_{jj} of happening, we assume that his estate will receive t_j/p_{jj} at the end of period j ; if he is alive at decision point $j + 1$, he will receive nothing. Since in this contract the mathematical expectation of “return” equals the cost, we shall say that the insurance is available at a “fair” rate. We assume that insurance is issued only when $p_{jj} < 1$, i.e. at decision points $1, \dots, J - 1$.

The purchase and sale of insurance on the lives of others will be viewed as a subset

¹ In real world situations, the individual would of course be forced to derive his own subjective probability distributions. Numerous descriptions of how this may be accomplished, on the basis of postulates presupposing certain consistencies in behavior, are available in the literature; see, for example, the accounts of Savage [23] and Marschak [16].

² The independence assumption, of course, is equivalent to the assumption that the state description of the economy is “fine” enough [17] to separate all the factors which cause dependence among investment returns over time.

of the productive opportunities. Each person will be assumed to give rise to a separate investment opportunity, the return of which is independent of the returns of all other opportunities.

2.2 The Utility Function

The amount spent on consumption in period j will be designated c_j . As indicated, c_j is a decision variable; in order to give it economic meaning, we require it to be non-negative. As noted earlier, it may also be interpreted as net capital withdrawals from the business. We assume that the amount to be consumed in period j is withdrawn at the beginning of the period.

We now postulate that the individual's preference ordering at the beginning of period j , conditioned on the event that death occurs in period $k \geq j$ and the economy is in states $n_j, n_{j+1}, \dots, n_{k+1}$ at decision points $j, j + 1, \dots, k + 1$, respectively, is representable by a numerical utility function $U_{jn_j \dots n_{k+1}}$ defined on the Cartesian product of all possible consumption programs (c_j, \dots, c_k) and the amount of the estate x'_{k+1} at the end of period k . We assume in this paper that the conditional utility function $U_{jn_j \dots n_{k+1}}$ has the form

$$\begin{aligned}
 & U_{jn_j \dots n_{k+1}}(c_j, \dots, c_k, x'_{k+1}) \\
 (5) \quad & = u(c_j) + \alpha_{jn_j n_{j+1}} u(c_{j+1}) + \dots + \alpha_{jn_j n_{j+1}} \dots \alpha_{k-1, n_{k-1} n_k} u(c_k) \\
 & \quad + \alpha_{jn_j n_{j+1}} \dots \alpha_{k-1, n_{k-1}, n_k} \delta_{kn_k n_{k+1}} g(x'_{k+1}). \quad j, k = 1, \dots, J, \quad j \leq k.
 \end{aligned}$$

We shall call $u(c)$ the on-period utility function of consumption and $g(x')$ the utility function of bequests. The constant $\alpha_{jmn} > 0$ is the patience factor linking the (one-period) utility functions of periods j and $j + 1$ given that the individual will be alive at decision point $j + 1$ and the economy moves from state m to state n in period j . When $\alpha_{jmn} < 1$ ($\alpha_{jmn} \geq 1$) we shall say that impatience (patience) prevails in period j , assuming a transition in that period from state m to state n , with respect to period $j + 1$ [13]. Similarly, the constant δ_{jmn} expresses the relative weight attached to bequests by the individual at decision point j given that death occurs in period j during a transition from state m to state n .

Implicit in form (5) is the assumption that preferences are independent over time for a given sequence of transitions of the economy. However, since transitions may affect investment returns as well as the *impatience rate*, we note that the preferences represented by the functions (5) are permitted to *depend on the opportunities* faced by the individual as well as (economic) conditions in general. This is in contrast to the usual assumption in economic models, which, as far as I am aware, has always separated preference from opportunity. An attempt to grapple with the question of allowing for flexibility of future preference has been made by Koopmans [12].

We also postulate that the individual always prefers more consumption to less in any period, i.e., that $u(c)$ is monotone increasing, and that the bequest function $g(x')$ is nondecreasing. In addition, it is assumed that the individual obeys the von Neumann-Morgenstern postulates³ [24] and that, accordingly, the utility functions (5) are cardinal and his objective is to maximize the expected utility attainable from consumption over his lifetime and the estate remaining and bequeathed upon his death. Finally, we postulate that the individual is a risk averter with respect to consumption, which im-

³ We assume, however, that the continuity postulate has been modified in such a way as to permit unbounded utility functions.

implies that $u(c)$ is strictly concave, and that $u(c)$ and $g(x')$ are twice differentiable. The objective function whose expectation with respect to $\{\beta_{ijmn}\}$ is to be maximized at decision point j may now be written $V_{jm}(c_j, x'_{j+1}, \dots, c_J, x'_{J+1})$, where

$$\begin{aligned}
 &V_{jm}(c_j, x'_{j+1}, c_{j+1}, \dots, c_J, x'_{J+1}) \\
 (5a) \quad &\equiv p_{jj} \sum_{n=1}^{N_{j+1}} p_{jmn} U_{jmn}(c_j, x'_{j+1}) + \dots + p_{jJ} \sum_{n=1}^{N_{j+1}} \dots \sum_{n_{J+1}=1}^{N_{J+1}} p_{jmn} p_{j+1, n n_1} \\
 &\quad \dots p_{J n_J n_{J+1}} U_{jmn \dots n_{J+1}}(c_j, \dots, c_J, x'_{J+1}), \quad j = 1, \dots, J, \quad \text{all } m.
 \end{aligned}$$

Clearly, the preference function V_{jm} depends on the state of the economy m which prevails at decision point j since both the transition probabilities p_{jmn} and the time preference rates α_{jmn} and δ_{jmn} , which in turn may depend on the opportunities F_{jmn} and r_{jm} , do. Thus we permit, in a very broad sense, the individual's preferences to be influenced by experience (specifically, the sequence of realizations of states of the economy) as it unfolds.⁴ In addition, the preferences that will prevail at decision points $j + 1, j + 2, \dots, J$ are clearly *stochastic* as of decision points $1, \dots, j$ since the states of the economy that will prevail in the future are known only probabilistically.

As far as I am aware, previous economic models have *in fact* assumed *basic* preferences to be strictly hereditary, occasionally retrospective (temporally nonseparable), but always unfazed by the environment. In the current model basic preferences are assumed to be shaped by both hereditary and environmental forces: the disposition toward risk (basically given by the shapes of the functions u and g) is determined by heredity alone, while the "discounting", both basic and effective, of future periods is traceable to both hereditary and environmental factors, as (5) and (5a) reveal.

3. Derivation of the Model

We shall now identify the relations which determine the amount of capital (debt) on hand at each decision point in terms of the amount on hand at the previous decision point. This leads to the pair of difference equations

$$(6) \quad x_{j+1, mn} = \sum_{i=2}^{M_{j+1}^m} \beta_{ijmn} z_{ij} + r_{jm} z_{1j}, \quad j = 1, \dots, J - 1, \quad \text{all } m, n$$

and

$$(7) \quad x'_{j+1, mn} = \sum_{i=2}^{M_{j+1}^m} \beta_{ijmn} z_{ij} + r_{jm} z_{1j} + t_j/p_{jj}, \quad j = 1, \dots, J, \quad \text{all } m, n$$

where

$$(8) \quad z_{ij} = x_j - c_j - t_j - \sum_{i=2}^{M_{j+1}^m} z_{ij}, \quad j = 1, \dots, J, \quad \text{all } m, n$$

by direct application of the definitions given in §2.1. The second and third subscripts to x_{j+1} indicate that the distribution of the random variable x_{j+1} may depend on the particular transition that took place in period j . These subscripts will only be used when necessary for clarity. The first terms of (6) and (7) represent the proceeds from productive investments, the second terms the payment of the debt or the proceeds from savings, and the third term in (7) the proceeds from life insurance.

Inserting (8) into (6) and (7) we obtain

$$\begin{aligned}
 (9) \quad x_{j+1, mn} &= \sum_{i=2}^{M_{j+1}^m} (\beta_{ijmn} - r_{jm}) z_{ij} + r_{jm} (x_j - c_j - t_j), \\
 & \quad j = 1, \dots, J - 1, \quad \text{all } m, n
 \end{aligned}$$

⁴ See paragraphs 2 and 3, §8 for the full implication of this statement.

and

$$(10) \quad x'_{j+1,mn} = \sum_{i=2}^{M_{jm}} (\beta_{ijmn} - r_{jm})z_{ij} + r_{jm}(x_j - c_j - t_j) + t_j/p_{jj},$$

$$j = 1, \dots, J, \quad \text{all } m, n.$$

The restriction that only the first S_{jm} opportunities may be sold short in period j when the economy is in state m at the beginning of that period implies that

$$(11) \quad z_{ij} \geq 0, \quad i = S_{jm} + 1, \dots, M_{jm}, \quad j = 1, \dots, J$$

must hold, while the assumption that all borrowing must be fully secured implies that x'_j must satisfy the conditions

$$(12) \quad \Pr \{x'_{jmn} \geq 0\} = 1, \quad m = 1, \dots, N_{j-1}, \quad n = 1, \dots, N_j, \quad j = 2, \dots, J + 1.$$

By (4) it follows that there is an upper limit on consumption in period $j, j = 1, \dots, J$, given by x_j , which, since $c_j \geq 0$, must be nonnegative for a feasible solution to exist in period j .

We shall now define $f_{jm}(x_j)$ as the maximum expected utility attainable by the individual over his remaining lifetime as of the beginning of period j on the condition that he is alive at that point, his capital is x_j , and the economy is in state m . Utilizing (5a), (1), and (5), we may write this definition formally:

$$(13) \quad \begin{aligned} f_{jm}(x_j) &\equiv \max \{E[V_{jm}(c_j, x'_{j+1}, c_{j+1}, \dots, x'_{J+1})] \mid x_j, \quad j = 1, \dots, J, \text{ all } m \\ &= \max E\{u(c_j) + p_{jj} \sum_{n=1}^{N_{j+1}} p_{jmn} \delta_{jmn} g(x'_{j+1,mn}) \\ &\quad + \sum_{k=j+1}^J p_{jk} \sum_{n=1}^{N_{j+1}} p_{jmn} \alpha_{jmn} u(c_{j+1}) \\ &\quad + p_{j,j+1} \sum_{n=1}^{N_{j+1}} \sum_{n_1=1}^{N_{j+2}} p_{jmn} p_{j+1,nn_1} \alpha_{jmn} \delta_{j+1,nn_1} g(x'_{j+2,nn_1}) \\ &\quad + \dots + p_{jj} \sum_{n=1}^{N_{j+1}} \dots \sum_{n_J=1}^{N_J} p_{jmn} \dots p_{J-1,n_{J-1}n_J} \alpha_{jmn} \\ &\quad \dots \alpha_{J-1,n_{J-1}n_J} (u(c_J) + \sum_{n_{J+1}=1}^{N_{J+1}} p_{Jn_Jn_{J+1}} \delta_{Jn_Jn_{J+1}} g(x'_{J+1,n_Jn_{J+1}})) \mid x_j, \\ &\quad j = 1, \dots, J, \quad \text{all } m. \end{aligned}$$

From (1) we obtain

$$(14) \quad p_{j,j+1}/(1 - p_{jj}) = \bar{p}_{j+1} \sum_{i=j}^J \bar{p}_i / \sum_{i=j}^J \bar{p}_i \sum_{i=j+1}^J \bar{p}_i = p_{j+1,j+1}$$

so that (13) may be written, by the principle of optimality,⁵

$$(15) \quad \begin{aligned} f_{jm}(x_j) &= \max \{u(c_j) + E[p_{jj} \sum_{n=1}^{N_{j+1}} p_{jmn} \delta_{jmn} g(x'_{j+1,mn}) \\ &\quad + (1 - p_{jj}) \sum_{n=1}^{N_{j+1}} p_{jmn} \alpha_{jmn} f_{j+1,n}(x_{j+1,mn})\} \mid x_j, \quad j = 1, \dots, J, \text{ all } m \end{aligned}$$

where

$$(16) \quad f_{J+1,m}(x_{J+1}) \text{ arbitrary } m = 1, \dots, N_{J+1}.$$

Dropping the symbol “ $\mid x_j$ ” and letting

$$(17) \quad \alpha_{jmn} \equiv p_{jj} \delta_{jmn}, \quad \text{all } j, m, n$$

and

⁵ The principle of optimality states that an optimal strategy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal strategy with regard to the state resulting from the first decision (see [2, p. 83]).

$$(18) \quad b_{jmn} \equiv (1 - p_{jj})\alpha_{jmn}, \quad \text{all } j, m, n$$

(15) may be written more concisely as

$$(19) \quad f_{jm}(x_j) = \max \{u(c_j) + \sum_{n=1}^{N_{j+1}} p_{jmn} E[a_{jmn} g(x'_{j+1,mn}) + b_{jmn} f_{j+1,n}(x_{j+1,mn})]\} \\ j = 1, \dots, J, \quad \text{all } m.$$

Note that $b_{Jmn} f_{J+1,n}(x_{J+1,mn})$ is identically zero since $p_{JJ} = 1$ by (1).

Utilizing (9), (10), (11) and (12), (19) becomes for all m
P1:

$$(20) \quad f_{jm}(x_j) = \max_{c_j, \{z_{ij}\}, t_j} \{u(c_j) + \sum_{n=1}^{N_{j+1}} p_{jmn} E[a_{jmn} g(\sum_{i=2}^{M_{j+1,m}} (\beta_{ijmn} - r_{jm}) z_{ij} \\ + r_{jm}(x_j - c_j - t_j) + t_j/p_{jj}) + b_{jmn} f_{j+1,n}(\sum_{i=2}^{M_{j+1,n}} (\beta_{ijmn} - r_{jm}) z_{ij} \\ + r_{jm}(x_j - x_j - t_j))]\}, \quad j = 1, \dots, J$$

subject to

$$(16) \quad f_{J+1,m}(x_{J+1}) \text{ arbitrary}$$

$$(21) \quad c_j \geq 0, \quad j = 1, \dots, J,$$

$$(11) \quad z_{ij} \geq 0, \quad i = S_{jm} + 1, \dots, M_{jm}, \quad j = 1, \dots, J,$$

$$(22) \quad t_j \geq 0, \quad j = 1, \dots, J - 1,$$

$$(22a) \quad t_J = 0$$

and

$$(23) \quad \Pr \{ \sum_{i=2}^{M_{j+1,m}} (\beta_{ijmn} - r_{jm}) z_{ij} + r_{jm}(x_j - c_j - t_j) + t_j/p_{jj} \geq 0 \} = 1, \\ n = 1, \dots, N_{j+1}, \quad j = 1, \dots, J.$$

In view of the fact that both the decision variables and the wealth component of the state description may vary continuously, both the state-space and the action-space are nondenumerable in the present model.

Since x represents capital, $f_{jm}(x)$ is clearly the utility of money at the j th decision point, given that the economy is in state m . Instead of being assumed, as is generally the case, the utility function of money has in this model been induced from inputs which are more basic than the preferences for money itself. As (20) shows, $f_{jm}(x)$ depends on the individual's preferences with respect to consumption and bequests, his age, his survival probabilities, the present state of the economy, the transition probabilities to and between states of the economy in the future, the available investment opportunities and their riskiness in the various states, as well as the interest rates in these states. Are not these the very factors that an individual, given the task of constructing his utility of money, would consider? Since money is only a means to an end, it should therefore come as no surprise that its utility is dependent on the utility of the end and the opportunities for achieving it.

We shall now attempt to obtain the solution to P1 for certain classes of the function $u(c)$ and the bequest function $g(x')$. More specifically, we shall consider the class of functions $u(c)$ such that $u(c)$ satisfies one (or more) of the functional equations

$$(24) \quad u(xy) = v(x)w(y),$$

$$(25) \quad u(xy) = v(x) + w(y)$$

for $c \geq 0$. The functional equations (24) and (25) in three unknowns belong to the set of equations usually referred to as the generalized Cauchy equations or Pexider's equations [1]. The subset of their solutions which is monotone increasing and strictly concave in u is given (leaving out v and w) (see [10]) by

- (26) $u(c) = c^\gamma, \quad 0 < \gamma < 1$ Model I
- (27) $u(c) = -c^{-\gamma}, \quad \gamma > 0$ Model II
- (28) $u(c) = \log c$ Model III.

Note that since $u(c)$ is a cardinal utility function, the solutions (26)–(28) also include every solution $\lambda_1 + \lambda_2 u(c)$ to (24) and (25) where λ_1 and $\lambda_2 > 0$ are constants, if simultaneously, $g(x')$ is represented by $\lambda_2 g(x')$.

In [10], it is also noted that (26)–(28) is the solution to the differential equation

$$(29) \quad cu''(c) + \gamma u'(c) = 0, \quad \gamma > 0.$$

Thus, (26)–(28) are also the only monotone increasing and strictly concave utility functions for which the proportional risk aversion index [21]

$$(30) \quad q^*(c) \equiv -cu''(c)/u'(c)$$

is a positive constant.

4. Preliminary Results

Before attempting to obtain the solution to P1, we shall state some preliminary results. For this purpose, let

$$(31) \quad \bar{v}_{jm} \equiv (v_{2jm}, \dots, v_{M_{jmn}}) \quad m = 1, \dots, N_j, \quad j = 1, \dots, J$$

be a vector of real numbers.

LEMMA. Let $\beta_{ijmn}, i = 2, \dots, M_{jm}$, and r_{jm} be defined as in §2. Then the set V_{jm} formed by the vectors \bar{v}_{jm} which satisfy the conditions

$$(32) \quad \Pr \left\{ \sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm})v_{ijm} + r_{jm} \geq 0 \right\} = 1, \quad n = 1, \dots, N_{j+1}$$

is nonempty, closed, convex, and bounded for all j and m .

PROOF. Let V_{jmn} denote the set of vectors \bar{v}_{jm} which satisfy

$$\Pr \left\{ \sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm})v_{ijm} + r_{jm} \geq 0 \right\} = 1.$$

Thus

$$(33) \quad V_{jm} = \bigcap_{n=1}^{N_{j+1}} V_{jmn}.$$

Let n be a state for which the “no-easy-money-condition” (4) holds. For such n , it was proved in [7] that V_{jmn} is nonempty, closed, convex, and bounded; for n such that (4) does not hold, V_{jmn} is also nonempty, closed, and convex, but not bounded. Thus, since $\bar{v}_{jm} = (0, \dots, 0) \in V_{jmn}$ for all n , it follows from (33) that V_{jm} is nonempty, closed, convex, and bounded, since the intersection of a finite number of closed, convex sets of which some are bounded is also closed, convex, and bounded.

THEOREM 1. Let $r_{jm}, \beta_{ijmn}, i = 2, \dots, M_{jm}, p_{jmn}$ and $u(c)$ be defined as in §2. Moreover, let $d_1, \dots, d_{N_{j+1}}$ be positive constants. Then each function h_{jm} given by

$$(34) \quad h_{jm}(\bar{v}_{jm}) \equiv \sum_{n=1}^{N_{j+1}} p_{jmn} d_n E[u(\sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm})v_{ijm} + r_{jm})]$$

subject to

$$(35) \quad v_{ijm} \geq 0, \quad i = S_{jm} + 1, \dots, M_{jm}$$

and

$$(32) \quad \Pr \left\{ \sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm})v_{ijm} + r_{jm} \geq 0 \right\} = 1, \quad n = 1, \dots, N_{j+1}$$

has a maximum and the maximizing \bar{v}_{jm} ($\equiv \bar{v}_{jm}^*$) is finite and unique.

PROOF. Let

$$(36) \quad h_{jmn}(\bar{v}_{jm}) \equiv E[u(\sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm})v_{ijm} + r_{jm})]$$

(34) may then be written

$$(37) \quad h_{jm}(\bar{v}_{jm}) = \sum_{n=1}^{N_{j+1}} p_{jmn} d_n h_{jmn}(\bar{v}_{jm}).$$

By the lemma, the set V_{jm} of vectors \bar{v}_{jm} which satisfy (32) is nonempty, closed, bounded and convex. The set V'_{jm} of vectors \bar{v}_{jm} which satisfy (35) is clearly also nonempty, closed, and convex. Thus, since $\bar{v}_{jm} = (0, \dots, 0)$ belongs both to V_{jm} and V'_{jm} , the set

$$(38) \quad D_{jm} \equiv V_{jm} \cap V'_{jm}$$

is nonempty, closed, convex, and bounded for all j and m .

We shall now show that (34) is strictly concave on D_{jm} for all j and m . To do this, we shall first establish that h_{jmn} is strictly concave on D_{jm} for all j, m , and n .

Consider points $\bar{v}_{jm}^1, \bar{v}_{jm}^2 \in D_{jm}$. By definition, h_{jmn} is strictly concave on D_{jm} if for all such points

$$(39) \quad h_{jmn}(\lambda \bar{v}_{jm}^1 + (1 - \lambda)\bar{v}_{jm}^2) > \lambda h_{jmn}(\bar{v}_{jm}^1) + (1 - \lambda)h_{jmn}(\bar{v}_{jm}^2)$$

whenever $0 < \lambda < 1$ and $\bar{v}_{jm}^1 \neq \bar{v}_{jm}^2$.

Let

$$(40) \quad \bar{w}_{jmn}^k \equiv \sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm})v_{ijm}^k + r_{jm}, \quad k = 1, 2.$$

The random variables \bar{w}_{jmn}^k clearly have a finite range since the β_{ijm} and r_{jm} have by assumption, and $\bar{v}_{jm}^k \in D_{jm}$, which is bounded. Using (36) and (40), the left-hand side of (39) gives

$$(41) \quad h_{jmn}(\lambda \bar{v}_{jm}^1 + (1 - \lambda)\bar{v}_{jm}^2) = E[u(\lambda \bar{w}_{jmn}^1 + (1 - \lambda)\bar{w}_{jmn}^2)].$$

The right side of (39) may be written, by (36) and (40),

$$(42) \quad E[\lambda u(\bar{w}_{jmn}^1)] + E[(1 - \lambda)u(\bar{w}_{jmn}^2)].$$

For every pair of values w_{jmn}^1 and w_{jmn}^2 of the random variables \bar{w}_{jmn}^1 and \bar{w}_{jmn}^2 such that \bar{v}_{jm}^1 and $\bar{v}_{jm}^2 \in D_{jm}$ we have, by the strict concavity of u ,

$$(43) \quad u(\lambda w_{jmn}^1 + (1 - \lambda)w_{jmn}^2) > \lambda u(w_{jmn}^1) + (1 - \lambda)u(w_{jmn}^2), \quad w_{jmn}^1 \neq w_{jmn}^2.$$

Consequently, when $\bar{v}_{jm}^1 \neq \bar{v}_{jm}^2 \in D_{jm}$ and $0 < \lambda < 1$, (43) implies that

$$(44) \quad \begin{aligned} E[u(\lambda \bar{w}_{jmn}^1 + (1 - \lambda)\bar{w}_{jmn}^2)] &> \lambda E[u(\bar{w}_{jmn}^1)] + (1 - \lambda)E[u(\bar{w}_{jmn}^2)] \\ &= \lambda h_{jmn}(\bar{v}_{jm}^1) + (1 - \lambda)h_{jmn}(\bar{v}_{jm}^2), \quad \text{all } j, m, n. \end{aligned}$$

Thus, by (41) and (44), (39) holds, i.e. h_{jmn} is strictly concave on D_{jm} for all n .

By assumption, we have

$$(45) \quad 0 \leq p_{jmn} d_n < \infty, \quad \text{all } j, m, n,$$

$$(46) \quad p_{jmn} d_n > 0, \quad \text{some } n, \text{ all } j, m.$$

Thus, $p_{jmn} d_n h_{jmn}$ is strictly concave on D_{jm} whenever (46) holds and is identically zero otherwise. Since the sum of (a finite number of) functions each of which is strictly concave on a set is also strictly concave on that set, it follows from (37) that h_{jm} is strictly concave on D_{jm} .

By the differentiability of u , it follows that h_{jm} is continuous on D_{jm} . Thus, since D_{jm} is closed, h_{jm} has a (finite) maximum on D_{jm} ; moreover, by the strict concavity of h_{jm} and the convexity of D_{jm} , \bar{v}_{jm}^* , the maximizing vector, is unique. This completes the proof.

COROLLARY 1. *Let $r_{jm}, \beta_{ijmn}, i = 2, \dots, M_{jm}, p_{jmn}$ and $u(c)$ be defined as in §2 and let $d_1, \dots, d_{N_{j+1}}$ be positive constants. Moreover, let $u(c)$ be such that it has no lower bound. Then the vector \bar{v}_{jm}^* which maximizes (34) subject to (35) and (32) is interior with respect to (32); that is*

$$(47) \quad \Pr \left\{ \sum_{i=2}^{M_{jm}} (\beta_{ijmn} - r_{jm}) v_{ijm}^* + r_{jm} > 0 \right\} = 1, \quad n = 1, \dots, N_{j+1}$$

for all j and m .

COROLLARY 2. *Let k_{jm} denote the maximum of function $h_{jm}(\bar{v}_{jm})$ in Theorem 2. Then*

$$(48) \quad k_{jm} > 0, \quad \text{all } j \text{ and } m \quad \text{Models I, III}$$

$$(49) \quad k_{jm} < 0, \quad \text{all } j \text{ and } m \quad \text{Model II.}$$

The proofs of the corollaries are trivial and will be omitted.

5. The Solution to P1

We shall now prove the following result.

THEOREM 2. *Let $a_{jmn}, b_{jmn}, p_{jmn}, \beta_{ijmn}, i = 2, \dots, M_{jm}, r_{jm}$, and F_{jmn} be defined as in §§2 and 3. Moreover, let $g(\cdot) = u(\cdot)$, let $u(c)$ be one of the functions (26), (27), or (28), and let*

$$(50) \quad a_{kmn}(1/p_{kk} - r_{km}) - b_{kmn} A_{k+1,n} r_{km} \leq 0, \\ n = 1, \dots, N_{k+1}, \quad k = j, \dots, J - 1, \quad \text{all } m$$

hold. Then a solution to P1 exists for $x_j \geq 0$ and is given, for $m = 1, \dots, N_j, j = 1, \dots, J$, by

$$(51) \quad f_{jm}(x_j) = A_{jm} u(x_j) + C_{jm}$$

$$(52) \quad c_{jm}^*(x_j) = B_{jm} x_j$$

$$(53) \quad z_{ijm}^*(x_j) = (1 - B_{jm}) v_{ijm}^* x_j, \quad i = 2, \dots, M_{jm}$$

$$(54) \quad z_{1jm}^*(x_j) = (1 - B_{jm})(1 - v_{1jm}^*) x_j$$

$$(55) \quad t_{jm}^* = 0$$

where the constants v_{ijm}^* ($v_{jm}^* \equiv \sum_{i=2}^{M_{jm}} v_{ijm}^*$) and k_{jm} are given by

$$k_{jm} \equiv \sum_{n=1}^{N_{j+1}} p_{jmn} (a_{jmn} + b_{jmn} A_{j+1,n}) E[u(\sum_{i=2}^{M_{jm}} (\beta_{ijmn} - r_{jm}) v_{ijm}^* + r_{jm})]$$

$$(56) \quad = \max_{\{v_{ijm}\}} \sum_{n=1}^{N_{j+1}} p_{jmn} (a_{jmn} + b_{jmn} A_{j+1,n}) \cdot E[u(\sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm}) v_{ijm} + r_{jm})]$$

subject to

$$(35) \quad v_{ijm} \geq 0, \quad i = S_{jm} + 1, \dots, M_{jm}$$

and

$$(32) \quad \Pr \{ \sum_{i=2}^{M_{jmn}} (\beta_{ijmn} - r_{jm}) v_{ijm} + r_{jm} \geq 0 \} = 1, \quad n = 1, \dots, N_{j+1}.$$

The constants A_{jm} , B_{jm} , and C_{jm} are given, for all m , by backward recursion on j from

(a) in the case of Model I

$$(57) \quad A_{jm} = (1 + k_{jm}^{1/(1-\gamma)})^{1-\gamma}$$

$$(58) \quad A_{J+1,m} \text{ arbitrary}$$

$$(59) \quad B_{jm} = (1 + k_{jm}^{1/(1-\gamma)})^{-1}$$

$$(60) \quad C_{jm} = 0$$

(b) in the case of Model II

$$(61) \quad A_{jm} = (1 + (-k_{jm})^{1/(1+\gamma)})^{1+\gamma}$$

$$(62) \quad A_{J+1,m} \text{ arbitrary}$$

$$(63) \quad B_{jm} = (1 + (-k_{jm})^{1/(1+\gamma)})^{-1}$$

$$(64) \quad C_{jm} = 0$$

(c) in the case of Model III

$$(65) \quad A_{jm} = 1 + \sum_{n=1}^{N_{j+1}} p_{jmn} (a_{jmn} + b_{jmn} A_{j+1,n})$$

$$(66) \quad A_{J+1,m} \text{ arbitrary}$$

$$(67) \quad B_{jm} = A_{jm}^{-1}$$

$$(69) \quad C_{jm} = k_{jm} - A_{jm} \log A_{jm} + (A_{jm} - 1) \log (A_{jm} - 1) + \sum_{n=1}^{N_{j+1}} p_{jmn} b_{jmn} C_{j+1,n}$$

$$(70) \quad C_{J+1,m} \text{ arbitrary.}$$

Furthermore, the solution is unique.

PROOF. We shall first consider the case where $j = J$. If $\delta_{Jmn} = 0$ for all m and n the proof for this case is trivial. Let us therefore turn to the case in which $\delta_{Jmn} > 0$ for some combinations of m and n . From P1 we then obtain, since $l_{Jm}^*(x_J) = 0$ by (22a) and $b_{Jmn} = 0$ by (18) and (1), for any m in the case of Model I,

$$(71) \quad f_{Jm}(x_J) = \max_{c_J, \{z_{iJ}\}} \{ c_J^\gamma + \sum_{n=1}^{N_{j+1}} p_{Jmn} a_{Jmn} E[(\sum_{i=2}^{M_{Jmn}} (\beta_{iJmn} - r_{Jm}) z_{iJ} + r_{Jm}(x_J - c_J))^\gamma] \}$$

subject to

$$(72) \quad c_J \geq 0$$

$$(73) \quad z_{iJ} \geq 0, \quad i = S_{Jm} + 1, \dots, M_{Jm}$$

and

$$(74) \quad \Pr \{ \sum_{i=2}^{M_{Jmn}} (\beta_{iJmn} - r_{Jm}) z_{iJ} + r_{Jm}(x_J - c_J) \geq 0 \} = 1, \quad n = 1, \dots, N_{J+1}.$$

By the “no-easy-money-condition” (4), it follows, since r_{Jm} is a positive constant, that (74) is satisfied, for any m , if and only if *either*

$$(75) \quad x_J - c_J > 0$$

and

$$(76) \quad \Pr \left\{ \sum_{i=2}^{M_{Jm}} (\beta_{iJmn} - r_{Jm})(z_{iJ}/(x_J - c_J)) + r_{Jm} \geq 0 \right\} = 1, \quad n = 1, \dots, N_{J+1}$$

or

$$(77) \quad x_J - c_J = 0$$

and

$$(78) \quad z_{iJ} = 0, \quad i = 2, \dots, M_{Jm}.$$

(72), (75), and (77) now imply that

$$(79) \quad x_J \geq 0$$

must hold for a solution to exist. By (75), (76), (77), and (78), (71) now becomes

$$(80) \quad f_{Jm}(x_J) = \max \{T_{Jm}(x_J), x_J^\gamma\}$$

where

$$(81) \quad T_{Jm}(x_J) = \sup_{c_J, \{z_{iJ}\}} \{c_J^\gamma + (x_J - c_J)^\gamma \sum_{n=1}^{N_{J+1}} p_{Jmn} a_{Jmn} \cdot E[(\sum_{i=2}^{M_{Jm}} (\beta_{iJmn} - r_{Jm})(z_{iJ}/(x_J - c_J)) + r_{Jm})^\gamma]\}$$

subject to (72), (73), (75) and (76).

Let

$$(82) \quad h_{Jm}(\bar{z}_J/(x_J - c_J)) \equiv \sum_{n=1}^{N_{J+1}} p_{Jmn} a_{Jmn} E[(\sum_{i=2}^{M_{Jm}} (\beta_{iJmn} - r_{Jm})(z_{iJ}/(x_J - c_J)) + r_{Jm})^\gamma].$$

Substituting (82) into (81) we get

$$(83) \quad T_{Jm}(x_J) = \sup_{c_J, \bar{z}_J} \{c_J^\gamma + (x_J - c_J)^\gamma h_{Jm}(\bar{z}_J/(x_J - c_J))\}$$

subject to the same constraints. Note that

$$(84) \quad \partial T_{Jm}/\partial h_{Jm} > 0$$

for all feasible c_J by (75). By Theorem 1 and (75), h_{Jm} subject to (73) and (76) has a (finite) maximum k_{Jm} (see (56)) and the maximizing vector

$$(85) \quad \bar{z}_J/(x_J - c_J) = \bar{v}_{Jm}^*$$

(see (56)) is unique for *every* c_J which satisfies (72) and (75). Thus, k_{Jm} is clearly *independent* of both c_J and x_J . Consequently, (83) becomes

$$(86) \quad T_{Jm}(x_J) = \sup_{0 \leq c_J < x_J} \{c_J^\gamma + (x_J - c_J)^\gamma k_{Jm}\}.$$

Solving

$$(87) \quad \partial T_{Jm}/\partial c_J = \gamma c_J^{\gamma-1} - \gamma k_{Jm} (x_J - c_J)^{\gamma-1} = 0$$

for c_J , we obtain

$$(88) \quad c_J = x_J / (1 + k_{Jm}^{1/(1-\gamma)}) = c_{Jm}^*(x_J).$$

Since

$$(89) \quad 0 < k_{Jm} < \infty$$

by Corollary 2 and Theorem 1, T_{Jm} is strictly concave on $[0, x_J]$, being the sum of two strictly concave functions (see [2:21]). Thus, $c_{Jm}^*(x_J)$ is unique. Furthermore, by (88) and (89), c_{Jm}^* is interior with respect to $[0, x_J]$ for $x_J > 0$, which guarantees that $T_{Jm}(x_J)$ is a maximum. Consequently, $T_{Jm}(x_J)$ is at least as great as x_J^γ for $x_J > 0$ so that (80) becomes for all m

$$(90) \quad f_{Jm}(x_J) = T_{Jm}(x_J).$$

Consequently, (88) is the optimal consumption strategy for $j = J$ and (85) gives, by (88) and (59),

$$(91) \quad z_{iJm}^*(x_J) = (1 - B_{Jm})v_{iJm}^*x_J, \quad i = 2, \dots, M_{Jm}$$

which is then also unique. The optimal lending strategy $z_{iJm}^*(x_J)$ now follows trivially from (8). Thus, upon insertion of the optimal strategies, (71) becomes

$$(92) \quad \begin{aligned} f_{Jm}(x_J) &= B_{Jm}^\gamma x_J^\gamma + k_{Jm}(x_J - B_{Jm}x_J)^\gamma \\ &= (1 + k_{Jm}^{1/(1-\gamma)})^{1-\gamma} x_J^\gamma \\ &= A_{Jm}x_J^\gamma \end{aligned}$$

which exists for $x_J \geq 0$. This concludes the proof for $j = J, m = 1, \dots, N_J$.

If A_{jm} , where j is one of the integers $1, \dots, J$, is a positive constant for all m , it follows from Theorem 1 and (56) that $k_{j-1,m}$ is a positive number for all m and that $v_{i,j-1,m}^*$ is finite and unique for all i and m . Since we have shown that A_{Jm} is a positive number for all m , we have that k_{jm} , as given by (56), is a positive constant for all j and m and that the $v_{i,jm}^*$ are finite and unique for all i, j , and m by (57), (61), and (65).

Now consider the case when $j = J - 1$. From P1 and (92) we then obtain for all m

$$(93) \quad \begin{aligned} f_{J-1,m}(x_{J-1}) &= \max_{c_{J-1}, t_{J-1}} \{c_{J-1}^\gamma + \sum_{n=1}^{N_J} p_{J-1,mn} \\ &\quad \cdot (a_{J-1,mn} E[(\sum_{i=2}^{M_{J-1,m}} (\beta_{i,J-1,mn} - r_{J-1,m}) z_{i,J-1} \\ &\quad + r_{J-1,m}(x_{J-1} - c_{J-1} - t_{J-1}) + t_{J-1}/p_{J-1,J-1})^\gamma] \\ &\quad + b_{J-1,mn} A_{Jn} E[(\sum_{i=2}^{M_{J-1,m}} (\beta_{i,J-1,mn} - r_{J-1,m}) z_{i,J-1} \\ &\quad + r_{J-1,m}(x_{J-1} - c_{J-1} - t_{J-1}))^\gamma])\} \end{aligned}$$

subject to

$$(94) \quad c_{J-1} \geq 0,$$

$$(95) \quad z_{i,J-1} \geq 0, \quad i = S_{J-1,m} + 1, \dots, M_{J-1,m},$$

$$(96) \quad t_{J-1} \geq 0,$$

$$(97) \quad \Pr \{ \sum_{i=2}^{M_{J-1,m}} (\beta_{i,J-1,mn} - r_{J-1,m}) z_{i,J-1} + r_{J-1,m}(x_{J-1} - c_{J-1} - t_{J-1}) + t_{J-1}/p_{J-1,J-1} \geq 0 \} = 1, \quad n = 1, \dots, N_J$$

and

$$(98) \quad \Pr \{ \sum_{i=2}^{M_{J-1,m}} (\beta_{i,J-1,mn} - r_{J-1,m}) z_{i,J-1} + r_{J-1,m}(x_{J-1} - c_{J-1} - t_{J-1}) \geq 0 \} = 1, \quad n = 1, \dots, N_J.$$

The constraints (98) follow from (79) and must be satisfied for the function f_{Jm} to exist, i.e. for the constraints (72) and (74) (which belong to (21) and (23)) to be satisfied.

It is immediately obvious that if (96) and (98) are satisfied, so is (97), since $p_{J-1,J-1} > 0$. Thus, (97) is redundant.

Since $r_{J-1,m}$ is a positive constant, it follows from the "no-easy-money-condition" (4) that, for any m , (98) is satisfied if and only if either

$$(99) \quad x_{J-1} - c_{J-1} - t_{J-1} > 0$$

and

$$(100) \quad \Pr \left\{ \sum_{i=2}^{M_{J-1,m}} (\beta_{i,J-1,mn} - r_{J-1,m}) \frac{z_{i,J-1}}{x_{J-1} - c_{J-1} - t_{J-1}} + r_{J-1,m} \geq 0 \right\} = 1, \\ n = 1, \dots, N_J$$

or

$$(101) \quad x_{J-1} - c_{J-1} - t_{J-1} = 0$$

and

$$(102) \quad z_{i,J-1} = 0, \quad \text{all } i.$$

Again, (94), (96), (99), and (101) imply that

$$(103) \quad x_{J-1} \geq 0$$

must hold for a solution to exist.

By (99), (100), (101), and (102), (93) becomes

$$(104) \quad f_{J-1,m}(x_{J-1}) = \max \{ T_{J-1,m}(x_{J-1}), \bar{T}_{J-1,m}(x_{J-1}) \}$$

where

$$(105) \quad \bar{T}_{J-1,m}(x_{J-1}) = \max_{0 \leq c_{J-1} \leq x_{J-1}} \{ c_{J-1}^\gamma + (x_{J-1} - c_{J-1})^\gamma \\ \cdot \sum_{n=1}^{N_J} p_{J-1,mn} a_{J-1,mn} (1/p_{J-1,J-1})^\gamma \}, \quad m = 1, \dots, N_{J-1}$$

and

$$(106) \quad T_{J-1,m}(x_{J-1}) = \{\text{right-hand side of (93)}\},$$

with "max" replaced by "sup", subject to (94), (95), (96), (99), and (100).

Taking the partial derivative of (106) with respect to t_{J-1} we obtain

$$(107) \quad \frac{\partial T_{J-1,m}}{\partial t_{J-1}} = \gamma \sum_{n=1}^{N_J} p_{J-1,mn} (a_{J-1,mn} (1/p_{J-1,J-1} - r_{J-1,m}) \\ \cdot E[(x_{Jmn} + t_{J-1}/p_{J-1,J-1})^{\gamma-1}] - b_{J-1,mn} A_{Jn} r_{J-1,m} E[x_{Jmn}^{\gamma-1}]).$$

When $t_{J-1} > 0$ and constraints (94), (95), (99), and (100) are satisfied, we have, since $p_{J-1,J-1}$ is positive by assumption,

$$(108) \quad (x_J + t_{J-1}/p_{J-1,J-1})^{\gamma-1} < x_J^{\gamma-1}$$

for every value the random variable x_J may assume which gives

$$(109) \quad E[(x_J + t_{J-1}/p_{J-1,J-1})^{\gamma-1}] < E[x_J^{\gamma-1}].$$

Thus, whenever (50) holds, we obtain

$$(110) \quad \partial T_{J-1,m} / \partial t_{J-1} \leq 0, \quad t_{J-1} \geq 0.$$

Moreover, by (94), (99), and (100), it is clear that both the set of feasible c_{j-1} and the set of feasible \bar{z}_{j-1} contract as t_{j-1} increases. Thus, whenever (50) holds, the optimal insurance strategy is given by

$$(111) \quad t_{j-1,m}^*(x_{j-1}) = 0, \quad \text{all } m$$

independently of c_{j-1} and \bar{z}_{j-1} .

(106) now becomes, for $m = 1, \dots, N_{j-1}$,

$$(112) \quad T_{j-1,m}(x_{j-1}) = \sup_{c_{j-1}, \bar{z}_{j-1}} \left\{ c_{j-1}^\gamma + (x_{j-1} - c_{j-1})^\gamma \sum_{n=1}^{N_j} p_{j-1,mn} (a_{j-1,mn} + b_{j-1,mn} A_{j_n}) \cdot E \left[\left(\sum_{i=2}^{M_{j-1,m}} (\beta_{i,j-1,mn} - r_{j-1,m}) \frac{z_{i,j-1}}{x_{j-1} - c_{j-1}} + r_{j-1,m} \right)^\gamma \right] \right\}$$

subject to (94), (95), (99), and (100) with $t_{j-1} = 0$. Letting

$$(113) \quad h_{j-1,m} \left(\frac{\bar{z}_{j-1}}{x_{j-1} - c_{j-1}} \right) \equiv \sum_{n=1}^{N_j} p_{j-1,mn} (a_{j-1,mn} + b_{j-1,mn} A_{j_n}) \cdot E \left[\left(\sum_{i=2}^{M_{j-1,m}} (\beta_{i,j-1,mn} - r_{j-1,m}) \frac{z_{i,j-1}}{x_{j-1} - c_{j-1}} + r_{j-1,m} \right)^\gamma \right]$$

we obtain, by the reasoning used for $j = J$, (56), (35), (32), and Theorem 1 that (112) becomes

$$(114) \quad T_{j-1,m}(x_{j-1}) = \sup_{0 \leq c_{j-1} < x_{j-1}} \{ c_{j-1}^\gamma + (x_{j-1} - c_{j-1})^\gamma k_{j-1,m} \} \geq \bar{T}_{j-1,m}(x_{j-1}), \quad m = 1, \dots, N_{j-1}.$$

(The inequality in (114) may be proved as follows: By (114) and (105), $T_{j-1,m}(x_{j-1}) \geq \bar{T}_{j-1,m}(x_{j-1})$ if and only if

$$(114a) \quad k_{j-1,m} \geq \sum_{n=1}^{N_j} p_{j-1,mn} a_{j-1,mn} (1/p_{j-1,j-1})^\gamma.$$

Since $\bar{z}_{j_m} = (0, \dots, 0)$ is feasible, (113) gives

$$(114b) \quad k_{j-1,m} \geq \sum_{n=1}^{N_j} p_{j-1,mn} (a_{j-1,mn} + b_{j-1,mn} A_{j_n}) r_{j-1,m}^\gamma.$$

By (50)

$$(114c) \quad a_{j-1,mn} (1/p_{j-1,j-1})^\gamma \leq (a_{j-1,mn} + b_{j-1,mn} A_{j_n})^\gamma r_{j-1,m}^\gamma \quad n = 1, \dots, N_j.$$

Since $0 < \gamma < 1$ and all constants in (114c) are positive, except $a_{j-1,mn}$ which is non-negative,

$$a_{j-1,mn} (1/p_{j-1,j-1})^\gamma \leq (a_{j-1,mn} + b_{j-1,mn} A_{j_n}) r_{j-1,m}^\gamma$$

which, by (114b), gives (114a).)

By (104) we obtain for all m ,

$$f_{j-1,m}(x_{j-1}) = \max_{0 \leq c_{j-1} \leq x_{j-1}} \{ c_{j-1}^\gamma + (x_{j-1} - c_{j-1})^\gamma k_{j-1,m} \}$$

which gives

$$f_{j-1,m}(x_{j-1}) = A_{j-1,m} x_{j-1}^\gamma$$

$$c_{j-1,m}^*(x_{j-1}) = B_{j-1,m} x_{j-1}$$

$$z_{i,J-1,m}^*(x_{J-1}) = (1 - B_{J-1,m})v_{i,J-1,m}^*x_{J-1} \quad i = 2, \dots, M_{J-1,m}$$

$$z_{1,J-1,m}^*(x_{J-1}) = (1 - B_{J-1,m})(1 - v_{J-1,m}^*)x_{J-1}.$$

For $j = 1, \dots, J - 2$, the proof is analogous to that for $j = J - 1$. In the case of Models II-III, the proof is similar and will therefore be omitted. Note, however, that by Corollary 1, the constraints (23) are never binding in the latter models (except for $j = J$ when $\delta_{Jmn} = 0$ for all n).

6. Properties and Implications of the Optimal Consumption Strategies

By reference to Theorem 2, we note that the optimal consumption function $c_{jm}^*(x_j)$ is proportional to capital x_j . For this reason, the optimal consumption strategy satisfies the permanent (normal) income hypotheses of Modigliani and Brumberg [19] and of Friedman [5], which form an important part of the so-called new consumption theories [3], precisely.

By reference to (59), (63), (67), (56), (48), and (49), we note that while the constant of (consumption) proportionality, B_{jm} , satisfies

$$(115) \quad 0 < B_{jm} \leq 1, \quad \text{all } j \text{ and } m, \quad \text{Models I-III}$$

where equality holds only for $j = J$ if $\delta_{Jmn} = 0$ for all n , it does not necessarily increase with age, i.e. in j . However, for "normal" values of the parameters representing the impatience to consume, the strength of the bequest motive, the survival probabilities, and the investment return distributions, we would generally expect to find a relatively large proportion of pairs (m, n) for which $p_{jmn} > 0$ such that

$$B_{jm} < B_{j+1,n}, \quad j = 1, \dots, J - 1.$$

Since

$$\partial B_{jm} / \partial b_{kmn} < 0, \quad j = 1, \dots, J - 1, \quad k = j, \dots, J - 1$$

whenever there is positive probability of going from state m in period j to state n in period $k + 1$, we obtain, using (18), that

$$\partial B_{jm} / \partial \alpha_{kmn} < 0, \quad j = 1, \dots, J - 1, \quad k = j, \dots, J - 1$$

under the same conditions. Thus, the greater an individual's impatience is with respect to some future period, the greater his present consumption would be—which is what we would expect.

By (30), we obtain

$$q^*(c) = 1 - \gamma, \quad q^*(c) = 1 + \gamma, \quad q^*(c) = 1$$

for Models I, II, and III, respectively. By examining the behavior of $\partial B_{jm} / \partial (1 - \gamma)$ and $\partial B_{jm} / \partial (1 + \gamma)$, respectively, it can be shown that it does not necessarily follow that the more risk averse an individual is, the less he will favor present consumption at the expense of future consumption.

A natural measure of the "favorableness" of the investment opportunities in the j th period, when the economy is in state m at the beginning of the period, is given by k_{jm} (see (56)). Since in the case of Model I

$$\partial B_{jm} / \partial k_{jm} < 0, \quad m = 1, \dots, N_j; \quad j = 1, \dots, J$$

while for Model II

$$(116) \quad \partial B_{jm} / \partial k_{jm} > 0, \quad m = 1, \dots, N_j; \quad j = 1, \dots, J,$$

we find that the propensity to consume in any period j and any state m is *decreasing* in k_{jm} in Model I and *increasing* in k_{jm} in Model II. In Model III, on the other hand, we observe from (67) that the optimal consumption strategy is *independent* of the investment opportunities in every respect.

The properties discussed so far do not depend explicitly on the Markovian character of the economic environment which we have postulated; they are quite similar to the properties exhibited by similar models developed in an environment characterized by stochastic independence between periods with regard to economic performance [8], [9]. We shall now examine some of the properties of the optimal consumption functions $c_{jm}^*(x_j)$ which are attributable to the Markovian nature of the preference functions and of the economic environment.

As stated in §2.2, the conditional functions (5) permit preferences to depend on opportunities, thus enabling our model to overcome the analytical separation of preference from opportunity employed in present economic theory. As a result, the actual preferences which will prevail at some time in the future will depend on the state (history) of the economy at that time (as yet unknown) as well as on the expectations of conditions from that point on.

The set of "relevant" states of the economy may of course vary from individual to individual. For example, if the entrepreneur is inclined to "keep up with the Joneses," he should allow, as a part of his description of the states of the economy, a listing of the possible things the Joneses might do which would affect his own preferences. This will in general greatly expand the number of possible states of the economy at a given decision point. However, if they are payoff-relevant [17], such states must clearly be distinguished since a distinction among them enables the individual to increase his total expected utility. This is borne out by the fact that the constants of (consumption) proportionality B_{jm} , for example, are not independent of the patience factors α_{jmn} corresponding to transitions from state m to state n at any decision point j , as can be seen from (56), (59), (63), and (67).

7. Properties of the Optimal Borrowing and Lending Strategies

Turning to (54), we find that the optimal lending strategy is linear in wealth $z_{1jm}^*(x_j)$, and hence lending, is clearly proportional to wealth if and only if $1 - v_{jm}^* > 0$ since $1 - B_{jm} > 0$ for all j , except $j = J$ when $\delta_{Jmn} = 0$ for all n , by (115). Analogously, $-z_{1jm}^*(x_j)$, or borrowing, is proportional to wealth if and only if $1 - v_{jm}^* < 0$. When $1 - v_{jm}^* = 0$, neither borrowing nor lending is optimal for any wealth level.

We shall now consider the case when the lending rate differs from the borrowing rate as is usually the case in the real world. Let $r_{jm}^L - 1$ and $r_{jm}^B - 1$ denote the lending and borrowing rates, respectively, in period j when the economy is in state m at the beginning of the period where

$$(117) \quad r_{jm}^L < r_{jm}^B, \quad \text{all } j, m.$$

Unfortunately, the sign of dv_{jm}^*/dr_{jm} is not readily determinable. However, since $\partial f_{jm}/\partial k_{im} > 0$, $i = j, \dots, J$, for all m and j in each model, the analysis is straightforward.

Let k_{jm}^L denote the maximum of (56) when the lending rate $r_{jm}^L - 1$ is used and the constraint

$$(118) \quad \sum_{i=2}^{M_{jm}} v_{ijm} \leq 1$$

is added to constraints (32) and (35). Since the set of vectors \bar{v}_{jm} which satisfy (118) is

convex and includes $\bar{v}_{jm} = (0, \dots, 0)$, Theorem 1 and the corollaries still hold when (118) is added to the constraint set. Furthermore, let k_{jm}^B denote the maximum of (56) under the borrowing rate $r_{jm}^B - 1$ subject to (32), (35) and

$$(119) \quad \sum_{i=2}^{M_{jm}} v_{ijm} \geq 1.$$

Again, Theorem 1 holds since the set of \bar{v}_{jm} which satisfies (119) is convex and any \bar{v}_{jm} such that $\sum_{i=2}^{M_{jm}} v_{ijm} = 1, v_{ijm} \geq 0$, satisfies all constraints. Setting

$$k_{jm} \equiv \max \{k_{jm}^L, k_{jm}^B\}, \quad \text{all } j, m$$

Theorem 2 holds as before. A “no-easy-money-condition” which takes the place of (4) when the borrowing rate exceeds the lending rate may be found in [8].

8. On the Optimal Investment Strategies

Proceeding to (53), we find that for any k, m , and j ,

$$(120) \quad z_{ijm}^*(x_j)/z_{kjm}^*(x_j) = v_{ijm}^*/v_{kjm}^*, \quad i = 2, \dots, M_{jm}.$$

Thus, it is apparent that in each period the optimal *mix* of risky (productive) investments in each of Models I-III is independent of the individual's wealth, but not the state of the economy, and that the optimal amount to invest in a given opportunity is proportional *in each state* to his capital x_j . By (53) and (56), it is also clear that the optimal investment strategy is *not* independent of the businessman's age, impatience to consume, strength of bequest motive, survival probabilities, and future investment opportunities, as is the case when preferences are independent of opportunities and investment returns are stochastically independent over time [8], [9].

By the structure of our model, we are permitting interest rates and investment returns to obey a Markov process which accounts for all changes in interest rates and all dependence *between periods* among investment returns. This is of course a significant advance from models in which interest rates are perfectly predictable and investment returns are stochastically independent over time. However, Markov processes are not generally veiwed as the most general of stochastic processes since they are defined as processes in which current probabilities depend only on the present state and not on previous states. In economics in particular, there is evidence that at least *recent* history also has some influence over current transition probabilities.

However, any stochastic process can in principle be reduced to a Markov process by an appropriate definition of the state space. Thus, if in the discrete-time process considered here the interest rate r_{jm} did not depend only on the state at the decision point j but the states of decision points $j - 10, j - 9, \dots, j - 1$ as well, “state” m would simply be viewed as a vector

$$m \equiv (m_{j-10}, m_{j-9}, \dots, m_{j-1}, m_j)$$

where m_j is a state distinguishable with respect to decision point j only. “State” m might then more appropriately be called history m ; p_{jmn} would then denote the probability of a transition to history n in period j given that the history at the beginning of j is m .

We now see that the model considered in this paper can cope with long-term investments whose returns obey the most general stochastic process in the context of stochastically constant returns to scale and perfect markets. Consider opportunity i in which we invested $z_{i,j-1,n}^*$ at the last decision point ($j - 1$) when the opportunity was brand new, say (and the economy in state n). If the economy is currently (decision

point j) in state m , this investment has a present market value of $\beta_{i,j-1,nm} z_{i,j-1,n}^*$. Thus,

$$(121) \quad z_{ijm}^* - \beta_{i,j-1,nm} z_{i,j-1,n}^*$$

clearly tells us how much we should presently *add* to our investment in opportunity i . If state m is favorable with respect to future returns from this opportunity, (121) is likely to be positive, calling for an increased commitment in this opportunity at the present time. However, if state m is unfavorable, (121) is probably negative (or zero), calling for partial or complete discontinuance of the investment. (121) of course represents the best adjustment *now* in light of all possible future developments and all future opportunities for adjustment of the level of commitment to opportunity i .

9. Insurance

When the businessman has no bequest motive, $\delta_{1n_1n_2}, \dots, \delta_{Jn_Jn_{J+1}} = 0$. This implies, as we have noted, that $B_{Jm} = 1$ for all m which gives

$$c_{Jm}^*(x_j) = x_j, \quad \text{all } m$$

$$z_{ijm}^*(x_j) = 0, \quad i = 1, \dots, M_{Jm}, \quad \text{all } m$$

and

$$\Pr \{x'_{J+1} = 0\} = 1,$$

i.e. the individual would consume all of his resources if he reaches decision point J .

From (4) and (10) we obtain that it is sufficient, but not necessary, that $z_{ijm}^*(x_j) \neq 0$ for some $i \geq 2$ in order that

$$\Pr \{x'_{j+1} > 0\} > 0, \quad j = 1, \dots, J - 1.$$

In fact, in Models II–III, it is sufficient by Corollary 1 for

$$\Pr \{x'_{j+1} > 0\} = 1, \quad j = 1, \dots, J - 1$$

that $x_j > 0$. Thus, should the individual pass away prior to the J th period, there is a very good chance that he will leave an estate even though he has no bequest motive. There are two factors which contribute to this, of course; first, his concern for future consumption should he be alive at the end of the current period, and second, the possibility that the return from his current investments will exceed the infimum return (Model I).

When $\delta_{jn_jn_{j+1}}, \dots, \delta_{Jn_Jn_{J+1}} = 0$, (50) is always satisfied which implies that the businessman should never buy insurance at a fair rate if he has no bequest motive. This is in contrast to the case of *some* individuals with exogenous noncapital income streams, for whom insurance purchases may be rational even though they lack a bequest motive [9].

The absence of a bequest motive (i.e. $\delta_{1n_1n_2}, \dots, \delta_{Jn_Jn_{J+1}} = 0$) is of course not necessary for (50) to hold, i.e. for noninsurance to be optimal. We see that for (50) to be violated, it is necessary for δ_{kmn} to be “large” relative to $b_{kmn}A_{k+1,n}$ for $k = j, \dots, J - 1$ and all m and n . However, since “large” $\delta_{k+1,n_{k+1},n_{k+2}}, \dots, \delta_{Jn_Jn_{J+1}}$ tend to make $A_{k+1,n}$ “large,” (50) is likely to be violated only if the bequest motive with respect to a given future period is unusually strong in comparison not only with the patience rates for the same period but also the bequest motives *and* the patience rates associated with subsequent periods. Thus, the condition which guarantees that the purchase of life insurance at fair rates is not optimal in the absence of an exogenous income stream can be expected to hold for most people who have no noncapital income stream.

So far we have assumed that the individual can only buy, and not sell, insurance on his own life. We shall now justify this assumption by showing that other individuals can be expected to be willing sellers, but not buyers, of insurance on his life.

Let us assume that the M_{jm} investment opportunities available in period j , given that the individual is in state m at the beginning of the period, have been indexed in such a way that the first $M'_{jm} \geq 2$ opportunities represent "regular" investments and that $i = M'_{jm} + 1, \dots, M_{jm}$ represent the individuals (other than himself) on whose lives the businessman has the opportunity to buy insurance. $\beta_{ijmn}, i = M'_{jm} + 1, \dots, M_{jm}$, will then assume, at the end of the period, the value 0 with probability $1 - p_{ij}$ or the value $1/p_{ij}$ with probability p_{ij} , where p_{ij} is the probability that individual i will pass away in period j (given that he is alive at decision point j). Thus, $E[\beta_{ijmn}] = 1, i = M'_{jm} + 1, \dots, M_{jm}$, for all m and n . Moreover, we assumed in §2.1 that the $\beta_{ijmn}, i = M'_{jm} + 1, \dots, M_{jm}$, are statistically independent of each other and of the other opportunities. Thus, we may write, upon denoting the right-most side of (56) by $h_{jm}(\bar{v}_{jm})$ and differentiating,

$$\frac{\partial h_{jm}}{\partial v_{ijm}}(v_{ijm} = 0) = \sum_{n=1}^{N_{j+1}} p_{ijn}(a_{jmn} + b_{jmn} A_{j+1,n}) E[u'(\sum_{i=2}^{M_{jm}} (\beta_{ijmn} - r_{jm})v_{ij} + r_{jm})]E[1 - r_{jm}]$$

$i = M'_{jm} + 1, \dots, M_{jm}, \text{ all } m < 0$

since $1 - r_{jm} < 0$ by assumption, the first expectation in each expression is positive by the monotonicity of u (and (32)), and the constants multiplying these expectations are positive by (48), (49), (57), (61), and (65). As a result, since h_{jm} is strictly concave in \bar{v}_{jm} by Theorem 1, $v_{ijm}^* \leq 0, i = M'_{jm} + 1, \dots, M_{jm}$, and inequality would hold whenever the solution \bar{v}_{jm}^* to (56) subject to (32) and (35) is interior with respect to (32). As shown by Corollary 1, this is always the case in Models II and III since they have no lower bound on u . Since $z_{ijm}^*(x_j) \leq 0$ whenever $v_{ijm}^* \leq 0$, an individual would thus never buy insurance on the life of another, thereby making it impossible for anyone to sell insurance on his own life. However, short sales, or simply sales, of insurance on the lives of others generally enable, as we have seen, the individual to increase his own utility since an increase in k_{jm} produces an increase in $f_{jm}(x_j)$. Thus, whenever an individual finds it rational to buy insurance on his own life at the postulated rate, he will always find willing suppliers of that insurance among those of his fellow men who obey one of Models I-III.

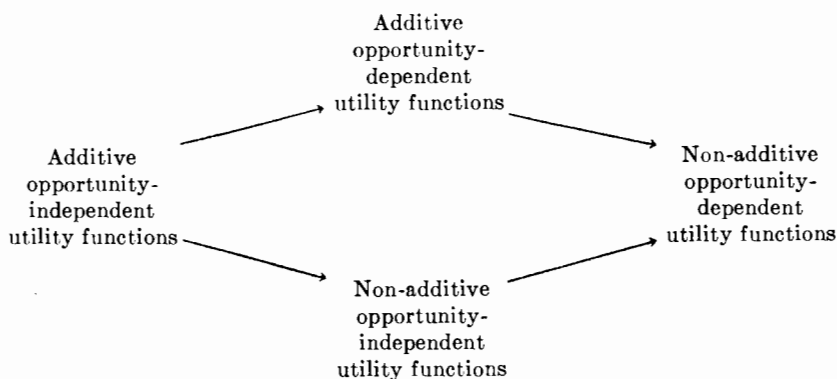
10. Concluding Remarks

We have in this paper examined a model of a firm with a single owner, generally known as a proprietorship. It might appear that the model could conceivably apply to multiperson firms such as corporations as well. In this case, the dividend policy would be represented by the consumption strategy, the capital budget by the investment strategies, and the financing policy, in the absence of equity financing, by the borrowing/lending strategy.

Suppose that a group of individuals have the same beliefs with respect to investment returns and the transitions of the states of the economy as well as the same disposition to risk, q^* , but differ in age, wealth, survival probabilities, impatience to consume, and strength of bequest motive. Thus, while the mix of consumption, all risky investments, and borrowing/lending for any individual is independent of wealth (see (52), (53), (54)), that mix is not independent of each individual's age, survival probabilities, im-

patience to consume, and strength of bequest motive, as noted in §§6–8. Consequently, there is no way a firm can serve individuals which differ in these personality characteristics *consistently with their objectives*, even if the firm is open-ended (in the sense that most mutual funds are open-ended; this feature, of course, enables each owner to draw a personalized dividend). This is in contrast to the situation in which the opportunities are stochastically independent over time. In this case, the mix of risky investments is independent not only of wealth but of age, survival probabilities, impatience to consume, strength of bequest motive, and in certain cases, exogenous noncapital income [6], [9], providing a basis for the formation and operation of firms with well-defined objective functions.

In this paper, we have employed a class of utility functions which is more general than utility functions which are additively separable in the sense that the utility functions considered are also dependent on opportunities. Another way to generalize additively separable utility functions is to consider utility functions which reflect the possibility that preferences concerning present and future consumption may depend on consumption levels actually experienced in the past. It is not clear which of these two generalizations is more significant empirically. However, judging from the results in the present paper, the introduction of state-dependence does offer promise of tractability. Obviously, opportunity-dependence is in a sense more “general” and in another sense less “general” than nonadditivity. A utility function more general than both of these must be dependent on both opportunities and past consumption. This gives the following lattice where \rightarrow is read “less general than.”



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